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6. AUTHOR(S)
Jason L. Speyer

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
University of California, Los Angeles
405 Hilgard Avenue
Los Angeles, CA 90024-1597

8. PERFORMING ORGANIZATION REPORT NUMBER

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9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
AFOSR
Attn: Dr. Marc Jacobs
Building 410
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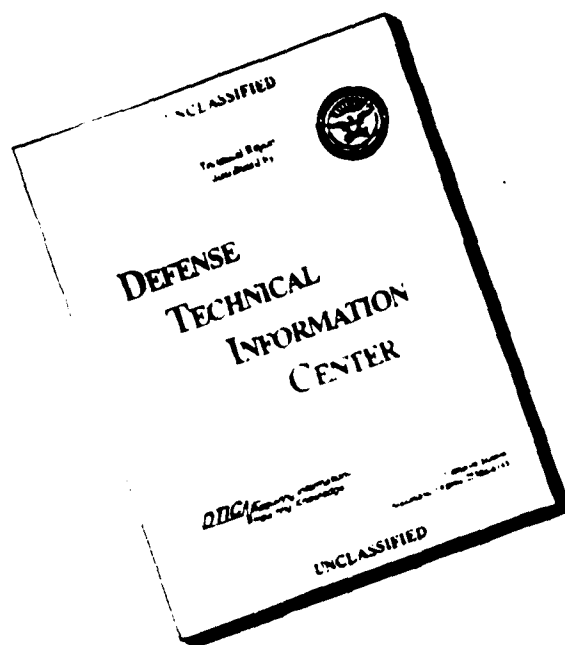| 17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
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Final Report to Air Force Office of Scientific Research

Mathematics Directorate

Robust and Adaptive Guidance and Control Laws for

Missile Systems

AFOSR 91-0077

Jason L. Speyer

Mechanical, Aerospace and Nuclear Engineering Department

University of California, Los Angeles

Los Angeles, California, 90024

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Contents

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Robust and Adaptive Guidance and Control Laws for Missile Systems

1 Objective of the Research Effort

The objective of this three year study is to develop robust and adaptive guidance and control laws for homing missiles, mechanizable with near-future computer technology, which can satisfy system objectives in the presence of large uncertainties and nonlinearities. Over the past years, considerable progress has been made in resolving some of the fundamental issues in homing guidance. Of particular importance, new filter structures, which were tailored to the passive homing engagement; and new target models and kinematic psuedo-measurements, which modified the new filter algorithm and induced a new adaptive homing guidance law, were developed. During the last three years in support of these important innovations, robust filters and control schemes which further enhance system performance were developed based upon a stochastic control problem known as the linear-exponential-Gaussian problem and a related deterministic approach called the disturbance attenuation problem. Most important, emerging from this work is a new structure for adaptive control and a unifying framework for developing midcourse and terminal homing missile guidance schemes under uncertainty.

2 Status of the Research Effort: Robust and Adaptive Guidance and Control Laws for Missile Systems

An active research program in systems theory has been established at UCLA in the Mechanical, Aerospace and Nuclear Engineering Department. One aspect of this research program centers on the development of robust high performance guidance and control schemes for such new concepts as bank-to-turn missile guidance systems. This activity considers new theoretical innovations and tailors them to missile system applications. Under this grant, the game theoretic approach to control and guidance syntheses, estimation with bearings-only measurements, and adaptive control were investigated. The missile system is an important conduit for motivating and using the results of this effort. Our approach is to develop realistic guidance systems based upon models of special dynamic and measurement systems. Previous results have been new state reconstruction algorithms based upon bearing-only measurements [1,2] and new missile guidance rules [3]. The missile guidance system presented in [3] represents the integration of our efforts over the years in estimation, target modeling and missile guidance laws. In this grant, robust filters and control schemes which further enhances system performance were developed based upon a stochastic control problem known as the linear-exponential-Gaussian problem and a related determistic approach called the disturbance attenuation problem. Emerging from this work is a new structure for adaptive control and a unifying framework for developing missile guidance schemes during midcourse and terminal homing under uncertainty.

2.1 Game Theoretic Synthesis

Since the game theoretic approach [4] is formulated in state space, it generalizes the currently popular H-infinity robustness techniques for time-invariant systems to finite-time and time-varying systems with both partial and full information. This is done by first defining a disturbance attenuation function as the ratio of an L_2 function of the outputs over an L_2 function of the disturbance inputs. A controller is to be determined which bounds

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the disturbance attenuation function in the presence of all input disturbances constrained to be in L_2. Furthermore, the game theoretic approach is related to the stochastic control problem of minimizing the expected value of an exponential of a quadratic argument subject to a Gauss-Markov process [5]. An essential feature of the solution to this so-called Linear-Exponential-Gaussian (LEG) problem is the realization that the expectation operator of an exponential form with respect to Gaussian random variables is equivalent to the extremization of the augmented quadratic argument formed from the exponential cost and the Gaussian probability density function. This leads to a game theoretic interpretation of this stochastic control problem. An application of the game theoretic approach to the problem of integration of the missile autopilot with the guidance system is given in [6].

2.2   Robust Game Theoretic Synthesis in the Presence of Uncertain Initial States and System Parameters

A game theoretic approach to linear control synthesis [4] is developed where initial states and parameter uncertainties are included as adversaries [7]. An implicit approach [8] closely related to μ-synthesis, characterizes the parameter dependence in the system coefficient matrices as linear. Although process and measurement uncertainties are included in the game cost criterion by quadratic penalty functions, the initial states and parameters are constrained to lie on or within given multi-dimensional ellipsoids. It is shown that the suboptimal H_∞ controller is not a saddle point strategy with this cost criterion even if the parameters are known. To solve this new dynamic game problem, a general linear control structure is assumed of given dimension for the output or partial information control problem so that the control is a function of a set of constant control parameters. In this game the adversaries are assumed to be knowledgeable about the value and the strategy of the control, although the controller is *only* aware of the measurement history. It is shown that under this circumstance, the dynamic game can be transformed into a parameter game problem between the control parameter set and plant unknown parameters if there exists an admissible control parameter set. If a saddle point strategy for the parameter game problem exists, then the saddle point

inequalities for the dynamic game problem are satisfied for process and measurement disturbances as well as for all initial states and parameters within their respective ellipsoids. This saddle point inequality guarantees a level of performance robustness as given by the value of the cost at the saddle point. In relation with the dynamic game problem, a disturbance attenuation problem is introduced where two types of disturbance attenuation parameters are used; one is associated with process and measurement disturbances and the other is associated with initial states. It is shown that these disturbance attenuation parameters are closely related to H_∞ norm and the cost criterion of the dynamic game.

2.3   A LEG Estimator

By probing further into the relationship between game theory and the stochastic LEG problem, very significant results have been obtained. By employing a worst-case performance measure, a game theoretic approach is used to determine a discrete-time state estimator [9,10]. The continuous estimator is reported in [11]. Although the order of minimization and maximization do not effect the saddle point value for this class of games, the order is critical in obtaining game theoretic strategies. *Two* interesting strategies result, an l_2 estimator which is the deterministic equivalent to the Kalman filter and the H_∞ estimator which satisfies a bound on a disturbance attenuation function. If a corresponding LEG problem is constructed where the quadratic argument of the exponential is the estimation error, then these two estimators under somewhat different assumptions still result and give the same saddle value of the augmented quadratic argument formed from the exponential. Note that it is well known (Sherman's Theorem) that minimizing, with respect to a function of the measurement history, the expectation of any symmetric, unimodal function of the estimation error (such as the exponential) subject to a Gauss-Markov system results in a conditional mean estimator, i.e., the Kalman filter. However, if the estimation error is replaced by the *sum* of estimation errors where these errors are functions of the measurement history up to the index of time in the sum, then Sherman's theorem does *not* hold, and the Kalman estimator is *not* minimizing. However, the stochastic LEG problem does have a unique solu-

tion which is the H_∞ estimator. This new filter generalizes the Kalman filter. Its statistical properties are now being investigated.

2.4   The Centralized and Decentralized LEG Problem

A third activity to be reported is that a uniform approach to the solution of the LEG problem constrained to the classical information pattern, one-step delayed information pattern, and one-step delay information-sharing pattern has been obtained [12]. The results vastly simplify the results of [13]. The one-step delayed information-sharing pattern allows a decentralized control structure where at each state of the dynamic programming algorithm a stochastic static team problem is solved. Furthermore, both convex and unimodal exponential functions are included which is an important extension of the work in [14]. Application of these results to sensor fusion in missile guidance systems should produce robust performance in the presence of electronic counter measures.

2.5   A Game Theoretic Dual Control Problem

Beginning with a disturbance attenuation function, a game is formulated for a special class of dual control problems. A scalar bilinear system is considered where the control coefficient is unknown and only an uncertain measurement of the state variable is available. The resulting controller, which is constrained to the measurement history, is a function of the state and parameter estimates and their associated pseudo-error variance. This controller depends upon the real roots of a fifth-order polynomial whose coefficients are also functions of both the estimates and pseudo-error variance. When the paper [15] was written, it was though that the controller had a dual control property because of the explicit appearance of the pseudo-variances. However, this is not the case for this formulation. Nevertheless, the resulting controller based only on the initial formulation as a disturbance attenuation problem did lead, without any approximations, to an interesting and mechanizable controller. This motivated new work in adaptive control discussed in the next subsection.

2.6 A Disturbance Attenuation Approach to Adaptive Control

In Rhee and Speyer [4], the disturbance attenuation problem is shown to extend the results of H_∞ analysis from time-invariant systems on infinite intervals to time-varying systems on finite intervals. By using a disturbance attenuation approach, it is attempted to limit the effect of any possible combination of disturbance and uncertainty to some small multiple of the disturbance. To this end, a disturbance attenuation function is constructed. The disturbance attenuation function is then converted to a performance index similar to those in more common optimal control problems. The problem then becomes a zero-sum game, in which the initial uncertainties and the system disturbances are considered as intelligent adversaries attempting to maximize the performance index, with the control playing their opponent, trying to minimize.

This approach has in the past resulted in very complex results when applied to other than a few special cases [15]. Speyer, Fruchter, and Hahn [15] applied it to a scalar system, and for a constant unknown control parameter, obtained a closed-form solution that hinged upon the solution of a fifth-order polynomial. In [16], an alternate approach which generalizes the results of Bernhard [17] appears to produce some simplification. By using the Principle of Optimality, the problem is split into two parts, each to be solved separately, and rejoined by an algebraic "connection" condition. The resulting problems are much simpler than the original, although in general still quite complex. In many cases, however, they can be reduced to a manageable level. In particular, for the case of constant parameters, with the system uncertainty limited to unknown control parameters, a solution suitable for real-time application is derived and the existance of a saddle point is shown.

The controller resulting from this approach is a function both of the state and parameter estimates and the associated curvature matrices which act as pseudo-variances. Although this approach represents a type of separation, no explicit assumption is made of certainty equivalence, as found in the development of current adaptive control algorithms.

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2.7 A System Characterization of Positive Real Conditions

Necessary and sufficient conditions for positive realness in terms of state space matrices are presented in [18] under the assumption of complete controllability and complete observability of square systems with ind pendent inputs. As an alternative to the positive real lemma and to the s-domain inequalities, these conditions provide a recursive algorithm for testing positive realness that result in a set of simple algebraic conditions. By relating the positive real property to an associated variational problem, the paper outlines a unified derivation of necessary and sufficient conditions for optimality of both singular and nonsingular problems. Based on this algorithm, a synthesis of a positive real system via output feedback is presented. This work is motivated by the need to determine cost criteria which allow control design based upon phase considerations rather than magnitude, the result of minimizing L_2 induces norms. Current work is in determining the characteristics of plants and compensators for which closed-loop transition matrix between desired output and disturbance inputs are positive real or dissipative.

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7. S. Hong, and J.L. Speyer, "Robust Game Theoretic Synthesis in the Presence of Uncertain Initial States and System Parameters," Proceeding of the 1992 Conference on Decision and Control.

8. I. Rhee and J.L. Speyer, "Application of a Game Theoretic Controller to a Benchmark Problem," *AIAA Journal of Guidance, Control and Dynamics*.

9. J.L. Speyer, R.N. Banavar, and C.H. Fan, "Linear Estimation with an Exponential Cost Criterion," 1992 IEEE Conference on Decision and Control.

10. R.N. Banavar and J.L. Speyer, "Risk-Sensitive Estimation and a Differential Game", to be published in the IEEE Transaction on Automatic Control

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3   Publications

1. I. Rhee and J.L. Speyer, "A Game-Theoretic Approach to a Finite-Time Disturbance Attenuation Problem", IEEE Transactions on Automatic Control, Vol. 36, No. 9, September 1991.

2. I. Rhee and J.L. Speyer, "Application of a Game Theoretic Controller to a Benchmark Problem", AIAA Guidance, Control, and Dynamics, Vol. 15, No. 5, September-October 1992.

3. H. Weiss, Q. Wang, and J.L. Speyer, "System Characterization of Positive Real Conditions", IEEE Transactions on Automatic Control, Vol. 39, No. 3, March 1994.

4. R.N. Banavar and J.L. Speyer, "Risk-Sensitive Estimation and A Differential Game", to be published in the IEEE Transaction in Automatic Control.

5. C.-H. Fan, J.L. Speyer and C.R. Jaensch, "Centralized and Decentralized Solutions of the Linear-Exponential Gaussian Problem", to be published in the IEEE Transaction in Automatic Control, November, 1994.

Appendix A contains these papers.

4 Research Professional Personnel

4.1 Professional Personnel

1. Chih-hai Fan

2. Sinpyo Hong

3. Ravi N. Banavar

4. David F. Chichka

5. Jinsheng Jang

4.2 Advanced Degrees Awarded

1. Chih-hai Fan, "Stochastic Control and Estimation with Exponential Cost Criteria", Doctoral Dissertation, 1992.

2. Ravi N. Banavar, "A Game Theoretic Approach to Linear Dynamic Estimation", Doctoral Dissertation, May 1992.

3. Sinpyo Hong, "Robust Game Theoretic Synthesis in the Presence of Uncertain Initial States and System Parameters", doctoral Dissertation, 1993.

5 Interactions

5.1 Papers in Proceedings of Technical Meetings

1. R.N. Banavar and J.L. Speyer, "A Linear-Quadratic Game Approach to Estimation and Smoothing", Proceedings of the 1991 American Controls Conference, June 1991.

2. S. Hong and J.L. Speyer, "Application of a Parameter Robust Game Theoretic Controller to the Benchmark Problem", Proceedings of the 1992 American Control Conference, June 24-26, 1992.

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ings of the 1992 American Control Conference, June 24-26, 1992.

4. C.-H. Fan, J.L. Speyer, C.R. Jaensch, "Decentralized Solutions of the Linear-Exponential-Gaussian Problem", Proceedings of the 31st IEEE Conference on Decision and Control, December 16-19, 1992.

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5.2   Interaction with Air Force Laboratories

During the grant period and also currently, we have been involved with Personnel at Eglin AFB on the implementation of nonlinear filters and target models, developed under AFOSR Sponsorship, to the AMRAAM missile product improvement program.

Appendix A: Papers published and Accepted for Publication

A Game Theoretic Approach to a Finite-Time Disturbance Attenuation Problem

Ihnseok Rhee and Jason L. Speyer, *Fellow, IEEE*

Abstract—A disturbance attenuation problem over a finite-time interval is considered by a game theoretic approach where the control, restricted to a function of the measurement history, plays against adversaries composed of the process and measurement disturbances, and the initial state. A zero-sum game, formulated as a quadratic cost criterion subject to linear time-varying dynamics and measurements, is solved by a calculus of variation technique. By first maximizing the quadratic cost criterion with respect to the process disturbance and initial state, a full information game between the control and measurement residual subject to the estimator dynamics results. The resulting solution produces an *n*-dimensional compensator which compactly expresses the controller as a linear combination of the measurement history. Furthermore, the controller requires the solution to two Riccati differential equations (RDE). For the linear saddle strategy of the controller necessary and sufficient conditions for the saddle point to be strictly concave with respect to all disturbances and initial conditions, and sufficient conditions for various process disturbance strategies to satisfy the saddle point condition are given. A disturbance attenuation problem is solved based on the results of the game problem. For time-invariant systems it is shown that under certain conditions the time-varying controller becomes time-invariant on the infinite-time interval. The resulting controller satisfies an H_∞ norm bound.

I. Introduction

RECENTLY, a time-domain control synthesis procedure for H_∞ control problems has been developed in [1], [3]. Riccati equations arising in the linear quadratic (LQ) game problem [12]–[14], [23] and Linear-Exponential-Gaussian (LEG) problem [15]–[18] play a key role in this synthesis procedure. These results motivated the formulation of a finite-time interval H_∞ control problem for time-varying systems based on a linear quadratic game approach. References [4], [5] considered a finite-time H_∞ control problem where the initial condition is given as zero. The full state information discrete-time H_∞ control problem is considered in [6], [7] by using existing linear quadratic game results.

In this paper, a finite-time interval disturbance attenuation problem for a time-varying system with uncertainty in the initial conditions of state is considered based on a LQ game theoretic formulation where the control plays against adversaries composed of the process and measurement disturbances and initial conditions. The general problem presented here is formulated as one of partial information such that the control is restricted to be a function of only the measurement history. A standard calculus variation procedure, which was shown in [13], [24] to be a useful tool for the full information LQ game problem, is adopted to solve the LQ game problem with partial information. The solution to this game problem generalizes many of the standard results given in [13], [14], [24]. The resulting compensator for the controller is *n*-dimensional requiring the solution to two Riccati differential equations (RDE). In particular, the solution is identical to that given by the continuous-time formulation of the Linear-Exponential-Gaussian (LEG) problem [16]. The formulation and solution of this finite-time time-varying game problem is given in Section III. By first maximizing the quadratic cost criterion with respect to the process disturbance and initial state, a full information game results between the control and measurement residual subject to the estimator dynamics. Three different saddle strategies for the process disturbance are considered and sufficient conditions for the existence of a saddle point solution are determined. In addition, necessary and sufficient conditions are determined for the saddle point to be strictly concave with respect to all nonzero variations of the disturbances and initial conditions from their saddle point strategies. Based on these results, conditions for the finite-time disturbance attenuation problem of Section IV are developed.

The finite-time solution is specialized in Section V to the infinite-time time-invariant solution based upon the two RDE's produced in Section III. In particular, it is shown in Section III-C that assuming the existence of a nonnegative definite solution to the algebraic Riccati equation (ARE), the solution to the RDE with certain initial conditions converges to the minimal nonnegative definite symmetric solution of the ARE. The results of Section III-C provide a proper development for the infinite-time time-invariant controller. In Section V it is shown that the *n*-dimensional compensators constructed from all the nonnegative solutions to two ARE, satisfying certain condition, satisfy the H_∞ norm bound.

Throughout this paper, $\| \cdot \|_A$ denotes the Euclidian norm weighted by A; $\mathcal{H}_x$ means $\partial \mathcal{H} / \partial x$; $D > 0 (\geq 0)$ means that D is a positive (nonnegative) definite matrix; $D > E (\geq E)$ means that $D - E > 0 (\geq 0)$.

During the course of the review process the authors become aware of similar results for the finite-time disturbance attenuation problem that were independently obtained in [8], [9].

II. Problem Statement

Consider a linear time-varying system described by

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + \Gamma(t)w(t) \tag{1}$$

$$z(t) = H(t)x(t) + \Gamma_1(t)v(t) \tag{2}$$

$$y(t) = \begin{bmatrix} C(t) \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ C_1(t) \end{bmatrix} u(t) \tag{3}$$

where x is the $n \times 1$ state vector; u is the $m \times 1$ input vector; z is the $p \times 1$ measurement vector; w and v are the $q \times 1$ and $p \times 1$ input disturbance vectors, respectively; y is the controlled output vector; and it is assumed that Γ_1 and $R = C_1^T C_1$ are nonsingular and the initial condition $x(0)$ is unknown. All matrices have appropriate dimensions and are time-varying.

Define the measurement history up to t as

$$Z_t = \{z(s), 0 \leq s \leq t\}.$$

The admissible control is restricted to be a function of only Z_t. Let $\mathcal{U}$ denote the set of admissible controls, and $\mathcal{U}_l \subset \mathcal{U}$ is the subset of linear functions of Z_t.

The disturbance attenuation problem to be considered is to find a control $u \in \mathcal{U}_l$ such that

$$\|x(T)\|_{Q_T}^2 + \int_0^T \|y\|^2 \, dt$$

$$< -\frac{1}{\theta} \left[\|x(0)\|_{P_0^{-1}}^2 + \int_0^T (\|w\|_{W^{-1}}^2 + \|v\|_{V^{-1}}^2) \, dt \right] \tag{4}$$

for all $w, v \in L_2[0, T]$, $x(0) \in R^n$ such that $(w(t), v(t), x(0)) \neq 0$ over all $t \in [0, T]$. It is assumed that θ is a negative constant, the final time is fixed, P_0, W, and V are time-varying positive definite matrices, and Q_T is a nonnegative definite matrix.

In order to solve the above problem, we will first consider as in [6], [7] the related linear quadratic game problem of finding $u^* \in \mathcal{U}$, $v^*, w^* \in L_2[0, T]$ and $x^*(0) \in R^n$ satisfying the saddle point condition

$$J(u^*, v, w, x(0)) \leq J(u^*, v^*, w^*, x^*(0))$$

$$\leq J(u, v^*, w^*, x^*(0)) \tag{5}$$

for all $u, v, w \in L_2[0, T]$, $x(0) \in R^n$, where

$$J(u, v, w, x(0)) = \frac{1}{2} \left[\frac{1}{\theta} \|x(0) - \hat{x}_0\|_{P_0^{-1}}^2 + \|x(T)\|_{Q_T}^2 \right.$$

$$\left. + \int_0^T \left\{ \|x\|_Q^2 + \|u\|_R^2 + \frac{1}{\theta}(\|w\|_{W^{-1}}^2 + \|v\|_{V^{-1}}^2) \right\} dt \right] \tag{6}$$

where $\hat{x}_0$ is a given vector and $Q = C^T C$. The left-hand side inequality plays an important role in the above disturbance rejection problem.

III. The Linear Quadratic Game Problem

A saddle point strategy can be obtained by solving two optimization problems

$$\min_{u \in \mathcal{U}} \max_v \max_w \max_{x(0)} J(u, v, w, x(0)) = J^* \tag{7}$$

$$\max_v \max_w \max_{x(0)} \min_{u \in \mathcal{U}} J(u, v, w, x(0)) = J_*. \tag{8}$$

The solutions to (7) and (8) produce saddle point strategies when $J^* = J_*$.

First, the optimization problem (7) is considered. By substitution of the constraint (2) into (6), we can change the optimization problem of (7) to the following problem:

$$\min_{u \in \mathcal{U}} \max_z \max_w \max_{x(0)} \frac{1}{2} \left[\frac{1}{\theta} \|x(0) - \hat{x}_0\|_{P_0^{-1}}^2 + \|x(T)\|_{Q_T}^2 \right.$$

$$+ \int_0^T \left\{ \|x\|_Q^2 + \|u\|_R^2 \right.$$

$$\left. \left. + \frac{1}{\theta}(\|w\|_{W^{-1}}^2 + \|(z - Hx)\|_{\bar{V}^{-1}}^2) \right\} dt \right] \tag{9}$$

subject to equation (1) where $\bar{V} = \Gamma_1 V \Gamma_1^T$. This cost criterion for the continuous deterministic game is remeniscent of the criterion constructed from the argument of the exponential in the discrete LEG problem [17], [18].

A. Maximization with Respect to w and $x(0)$

To solve the problem consider first the maximization of J with respect to w and $x(0)$ for a given z and fixed strategy $u \in \mathcal{U}$ for which the variations of u and z vanish. The resulting cost criterion will then be minimized and maximized with respect to u and z, respectively. Let

$$J_1 = \max_w \max_{x(0)} J.$$

The standard variational procedure [24] is formally applied to this problem. A vector Lagrange multiplier function λ is introduced to adjoin (1) to (9). The first variation of J for given u and z is given by

$$\delta J = \frac{1}{\theta} \left[P_0^{-1} \{x(0) - \hat{x}_0\} + \lambda(0) \right]^T \delta x(0)$$

$$+ \left[Q_T x(T) - \frac{1}{\theta} \lambda(T) \right]^T \delta x(T)$$

$$+ \int_0^T \left[\frac{1}{\theta} (\dot{\lambda}^T + \theta \mathcal{H}_x) \delta x + \mathcal{H}_w \, \delta w \right] dt$$

where $\mathcal{H}$ is the Hamiltonian, defined by

$$\mathcal{H}(x, w, \lambda) = \frac{1}{2} \left\{ x^T Q x + u^T R u + w^T (\theta W)^{-1} w \right.$$

$$+ (z - Hx)^T (\theta \bar{V})^{-1} (z - Hx) \}$$

$$+ \frac{1}{\theta} \lambda^T (Ax + Bu + \Gamma w).$$

The first-order necessary conditions for a maximum are

$$\dot{\lambda} = -\theta \mathcal{H}_x^T, \quad \lambda(T) = \theta Q_T x(T),$$

$$\lambda(0) = -P_0^{-1} \{x(0) - \hat{x}_0\} \tag{10}$$

$$\mathcal{H}_w = 0 \Rightarrow w = -W \Gamma^T \lambda. \tag{11}$$

After substituting (11) into (1) the following two point

boundary value problem is obtained:

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -\Gamma W \Gamma^T \\ -(\theta Q + H^T \bar{V}^{-1} H) & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}$$
$$+ \begin{bmatrix} Bu \\ H^T \bar{V}^{-1} z \end{bmatrix} \quad (12)$$

with

$$x(0) = \hat{x}_0 - P_0 \lambda(0), \quad \lambda(T) = \theta Q_T x(T). \quad (13)$$

Since the two point boundary value problem is linear, the solution can be obtained by the sweep method [24]. Let x_* and λ_* denote the solutions to (12) and (13). We assume that the solution x_* satisfies

$$x_* = \hat{x} - P \lambda_* \quad (14)$$

where $\hat{x}$ and P are to be determined. The substitution of the differentiation of (14) into (12) yields

$$\{ \dot{P} - PA^T - AP + P(H^T \bar{V}^{-1} H + \theta Q)P - \Gamma W \Gamma^T \} \lambda_*$$
$$= \dot{\hat{x}} - A\hat{x} - Bu - PH^T \bar{V}^{-1}(z - H\hat{x}) + \theta PQ\hat{x}.$$

Therefore, if we choose $\hat{x}$ and P to satisfy the following:

$$\dot{\hat{x}} = A\hat{x} + Bu + PH^T \bar{V}^{-1}(z - H\hat{x}) - \theta PQ\hat{x},$$
$$\hat{x}(0) = \hat{x}_0 \quad (15)$$

$$\dot{P} = PA^T + AP - P(H^T \bar{V}^{-1} H + \theta Q)P + \Gamma W \Gamma^T$$
$$P(0) = P_0 \quad (16)$$

then (14) becomes an identity. From (13) and (14) we obtain

$$\lambda_*(T) = \theta Q_T \{ I + \theta P(T) Q_T \}^{-1} \hat{x}(T) \quad (17)$$

$$x_*(T) = \{ I + \theta P(T) Q_T \}^{-1} \hat{x}(T) \quad (18)$$

where we assume $\{ I + \theta P(T) Q_T \}$ is nonsingular. Then, we can calculate $\lambda_*(t)$ from

$$\dot{\lambda}_* = -[A - P(H^T \bar{V}^{-1} H + \theta Q)]^T \lambda_*$$
$$+ H^T \bar{V}^{-1}(z - H\hat{x}) - \theta Q\hat{x} \quad (19)$$

with the final condition given by (17).

The second variation is considered to determine additional necessary conditions of optimality. The second variation along the extremal path $\delta^2 J$ is given as

$$\delta^2 J = \delta x(0)^T (\theta P_0)^{-1} \delta x(0) + \delta x(T)^T Q_T \delta x(T)$$
$$+ \frac{1}{\theta} \int_0^T [\delta x^T (H^T \bar{V}^{-1} H + \theta Q) \delta x + \delta w^T W^{-1} \delta w] \, dt. \quad (20)$$

From the constraint (1), we obtain

$$\delta \dot{x} = A \, \delta x + \Gamma \, \delta w. \quad (21)$$

Adding the zero quantity

$$0 = [\delta x^T (\theta P)^{-1} \delta x]_0^T - \int_0^T \frac{d}{dt} \{ \delta x^T (\theta P)^{-1} \delta x \} \, dt$$

to (20) and using (21), we obtain the second variation $\delta^2 J$ as a perfect square of the form

$$\delta^2 J = \frac{1}{\theta} \Big[\| \delta x(T) \|_{P^{-1}(T) + \theta Q_T}^2$$
$$+ \int_0^T \| \delta w - W \Gamma^T P^{-1} \delta x \|_{W^{-1}}^2 \, dt \Big]$$

where we assume $P(t)$ is nonsingular over $t \in [0, T]$. Since the performance index is quadratic, for a maximum the second variation should be negative for all variations $\delta x(0)$ and δw not vanishing simultaneously. Consider the variation

$$\delta w = W \Gamma^T P^{-1} \delta x. \quad (22)$$

Then, from (21)

$$\delta \dot{x} = (A - \Gamma W \Gamma^T P^{-1}) \delta x. \quad (23)$$

Hence

$$\delta x(T) = \Phi(T, 0) \delta x(0)$$

where Φ denotes the transition matrix of the linear differential equation (23). Thus

$$\delta^2 J = \frac{1}{\theta} \delta x(0)^T \Phi^T(T, 0) \{ P^{-1}(T) + \theta Q_T \} \Phi(T, 0) \delta x(0)$$
$$< 0, \quad \forall \delta x(0) \neq 0$$

which implies that for a maximum it is necessary that

$$P^{-1}(T) + \theta Q_T > 0 \quad (24)$$

since $\Phi(T, 0)$ is nonsingular. Also, (24) implies $P(T) > 0$ since $Q_T \geq 0$.

We will show that if the condition (24) holds and P^{-1} is finite, the second variation is negative for all variation $\delta x(0)$ and δw not vanishing simultaneously. If (24) is satisfied, then for all variations

$$\delta^2 J \leq 0$$

where the equality holds for the variation given in (23) and

$$\delta x(T) = 0. \quad (25)$$

However, for the variation (22) and (25), (23) leads to $\delta x(t) = 0$ and $\delta w(t) = 0$ over $t \in [0, T]$ if P^{-1} is finite. That is, $w_*(t) = -W \Gamma^T \lambda_*(t)$ and $x_*(0) = \hat{x}_0 - P_0 \lambda_*(0)$ produce the maximum.

B. The Solution to Problem (7)

To perform the minimization and maximization with respect to u and z, first evaluate J_1. By using (10), (11), and (14), J_1 can be represented as

$$J_1 = \frac{1}{2\theta} [\| \lambda_*(0) \|_{P_0}^2 + \| x_*(T) \|_{(\theta Q_T)}^2]$$
$$+ \frac{1}{2} \int_0^T \Big[\| \hat{x} - P\lambda_* \|_Q^2 + \| u \|_R^2$$
$$+ \frac{1}{\theta} \{ \| \lambda_* \|_{\Gamma W \Gamma^T}^2 + \| z - H(\hat{x} - P\lambda_*) \|_{\bar{V}^{-1}}^2 \} \Big] \, dt.$$

Adding the zero quantity

$$0 = \frac{1}{2\theta} [\lambda_*^T P \lambda_*]_0^T - \frac{1}{2\theta} \int_0^T \frac{d}{dt} (\lambda_*^T P \lambda_*) \, dt$$

to J_1 yields (using (16) and (19))

$$J_1 = \frac{1}{2} x_*(T)^T Q_T x_*(T) + \frac{1}{2\theta} \lambda_*(T)^T P(T) \lambda_*(T)$$

$$+ \frac{1}{2} \int_0^T \Big\{ \hat{x}^T Q \hat{x} + u^T R u$$

$$+ (z - H\hat{x})^T (\theta \overline{V})^{-1} (z - H\hat{x}) \Big\} \, dt.$$

From (17) and (18)

$$x_*(T)^T Q_T x_*(T) + \frac{1}{\theta} \lambda_*(T)^T P(T) \lambda_*(T)$$

$$= \hat{x}(T)^T S_T \hat{x}(T) \quad (26)$$

where

$$S_T = Q_T \{ I + \theta P(T) Q_T \}^{-1}.$$

By the matrix inversion lemma S_T becomes

$$S_T = Q_T - \theta Q_T \{ P^{-1}(T) + \theta Q_T \}^{-1} Q_T.$$

Therefore, condition (24) implies that $S_T \geq 0$. Now, by using (26) J_1 can be represented in terms of $\hat{x}$ as

$$J_1 = \frac{1}{2} \hat{x}(T)^T S_T \hat{x}(T) + \frac{1}{2}$$

$$\cdot \int_0^T \Big\{ \hat{x}^T Q \hat{x} + u^T R u + (z - H\hat{x})^T (\theta \overline{V})^{-1} (z - H\hat{x}) \Big\} \, dt.$$

The minimization and maximization can be performed with respect to u and z, respectively, subject to (15). By defining $\hat{v} \triangleq z - H\hat{x}$, this problem can be changed to

$$\min_u \max_{\hat{v}} \frac{1}{2} \Big[\hat{x}(T)^T S_T \hat{x}(T)$$

$$+ \int_0^T \Big\{ \hat{x}^T Q \hat{x} + u^T R u + \hat{v}^T (\theta \overline{V})^{-1} \hat{v} \Big\} \, dt \Big] \quad (27)$$

subject to

$$\dot{\hat{x}} = \overline{A} \hat{x} + Bu + \overline{\Gamma} \hat{v}, \quad \hat{x}(0) = \hat{x}_0 \quad (28)$$

where $\overline{A} \triangleq A - \theta PQ$ and $\overline{\Gamma} \triangleq PH^T \overline{V}^{-1}$. We obtain (28) from (15) by using the definition of $\hat{v}$. Finally, the problem reduces to the well-known deterministic game problem. The optimal feedback strategies $u^* \in \mathcal{U}$ and $\hat{v}^*$ for (27) are given in [14], [23] as

$$u^* = -R^{-1} B^T S \hat{x} \quad (29)$$

$$\hat{v}^* = -\theta \overline{V} \overline{\Gamma}^T S \hat{x} = -\theta HPS \hat{x} \quad (30)$$

if the Riccati equation

$$-\dot{S} = S\overline{A} + \overline{A}^T S - S(BR^{-1}B^T + \theta \overline{\Gamma} \overline{V} \overline{\Gamma}^T)S + Q$$

$$= S(A - \theta PQ) + (A - \theta PQ)^T S$$

$$- S(BR^{-1}B^T + \theta PH^T \overline{V}^{-1} HP)S + Q \quad (31)$$

with the terminal condition $S(T) = S_T$ has a solution over $t \in [0, T]$. It is noted that open-loop strategies for u may make the cost (27) unbounded without certain condition while the feedback strategies (29) makes the cost bounded (see [23]). The cost criterion using the u^* and $\hat{v}^*$ is given as

$$J_1(u^*, \hat{v}^*) = \frac{1}{2} \hat{x}_0^T S(0) \hat{x}_0. \quad (32)$$

Let $x^*(t)$, $\lambda^*(t)$, and $\hat{x}^*(t)$ denote the optimal trajectory of x, λ, and $\hat{x}$, respectively. $x^*(t)$, $\lambda^*(t)$, and $\hat{x}^*(t)$ are solutions of x_*, λ_*, and $\hat{x}$ with $u = u^*$ and $\hat{v} = \hat{v}^*$. Substituting (29) and (30) into (15) and (19) yields

$$\dot{\hat{x}}^* = (A - \theta PQ) \hat{x}^* - BR^{-1}B^T S \hat{x}^*$$

$$- \theta PH^T \overline{V}^{-1} HPS \hat{x}^*, \quad \hat{x}^*(0) = \hat{x}_0 \quad (33)$$

$$\dot{\lambda}^* = -(A - \theta PQ)^T \lambda^* + H^T \overline{V}^{-1} HP(\lambda^* - \theta S \hat{x}^*)$$

$$- \theta Q \hat{x}^* \quad (34)$$

with $\lambda^*(T) = \theta S_T \hat{x}^*(T)$. $\hat{x}^*(t)$ can be calculated from (33) independently. Observe that

$$\frac{d}{dt} (\theta S \hat{x}^*) = -(A - \theta PQ)^T (\theta S \hat{x}^*) - \theta Q \hat{x}^*. \quad (35)$$

Comparing (34) and (35) gives

$$\lambda^*(t) = \theta S \hat{x}^*(t), \quad t \in [0, T]. \quad (36)$$

Substituting (29) and (36) into (12) and (14) yields

$$\dot{x}^* = Ax^* - (BR^{-1}B^T S + \theta \Gamma W \Gamma^T S) \hat{x}^*,$$

$$x^*(0) = \{ I - \theta P_0 S(0) \} \hat{x}_0.$$

Observe that

$$\frac{d}{dt} [(I - \theta PS) \hat{x}^*] = A(I - \theta PS) \hat{x}^*$$

$$- (BR^{-1}B^T S + \theta \Gamma W \Gamma^T S) \hat{x}^*.$$

By inspection we obtain the optimal state trajectory as

$$x^*(t) = (I - \theta PS) \hat{x}^*(t), \quad t \in [0, T]. \quad (37)$$

From (2), (30), (37) and the definition of $\hat{v}$, the optimal trajectory for v and $x(0)$, denoted as $v^*(t)$ and $x^*(0)$, respectively, are given as

$$v^*(t) = \Gamma_1^{-1} \{ \hat{v}^*(t) + H(\hat{x}^*(t) - x^*(t)) \}, \quad t \in [0, T]$$

$$= 0, \quad (38)$$

$$x^*(0) = \{ I - \theta P_0 S(0) \} \hat{x}_0. \quad (39)$$

From (11), (36), and (37), we obtain the optimal trajectory for w, denoted as $w^*(t)$, in terms of $\hat{x}^*(t)$ as

$$w^*(t) = -\theta W \Gamma^T S \hat{x}^*(t), \quad t \in [0, T] \quad (40)$$

or in terms of $x^*(t)$, by assuming $I - \theta PS$ is nonsingular[1] over $[0, T]$, as

$$w^*(t) = -\theta W \Gamma^T \Pi x^*(t), \quad t \in [0, T] \quad (41)$$

[1] Conditions that ensure this inverse are given in the next section.

where

$$\Pi = S(I - \theta PS)^{-1}. \tag{42}$$

Equations (38), (39), (40), and (41) denote the open-loop strategies for v, $x(0)$ and w. The open-loop strategy for u is obtained by substituting $\hat{x}^*$ for $\hat{x}$ in (29). However, since the cost (27) may become unbounded for the open-loop strategy for u, the closed-loop strategy confined to be a function of the measurement history (29) should be used. Sufficient conditions for the existence of various saddle point strategies are given in Section III-D. The cost along the u^*, v^*, w^*, and $x^*(0)$ is obtained from (32) as

$$J^* = J(u^*, v^*, w^*, x^*(0)) = \frac{1}{2}\hat{x}_0^T S(0)\hat{x}_0. \tag{43}$$

C. Some Properties of Riccati Equations

Consider a Riccati differential equation of the form

$$-\dot{X} = \mathcal{A}^T X + X\mathcal{A} - X(\mathcal{B}\mathcal{B}^T - \mathcal{D}\mathcal{D}^T)X + \mathcal{Q},$$
$$X(T) = X_T \geq 0, \quad t \leq T \tag{44}$$

where the coefficient matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{Q}$ are time-varying matrices. It is assumed that $\mathcal{Q}$ and X_T are nonnegative definite matrices. Note that the RDE (16) has this form if we change independent variable from t to $\tau = T - t$.

Let $X(t, X_T)$ denote the solution to the RDE (44), if it exists.

Lemma 1: Suppose that $X(t, X_T)$ exists over $[t_0, T]$. If $X_T \geq 0(> 0)$, then $X(t, X_T)$ is nonnegative (positive) definite over $[t_0, T]$.

Proof: Suppose that $X(t, X_T)$ is a solution to (44) with $X_T \geq 0(> 0)$ over $[t_0, T]$. Then, $X(t, X_T)$ satisfies the Riccati differential equation

$$-\dot{X} = \mathcal{A}^T X + X\mathcal{A} - X\mathcal{B}\mathcal{B}^T X + \bar{\mathcal{Q}}, \quad X(T) = X_T \tag{45}$$

for all $t \in [t_0, T]$ where

$$\bar{\mathcal{Q}}(t) \triangleq \mathcal{Q} + X(t, X_T)\mathcal{D}\mathcal{D}^T X(t, X_T).$$

Note that $\bar{\mathcal{Q}}(t)$ is nonnegative definite over $[t_0, T]$. Hence, [22, Theorem 2.1] shows that $X(t, X_T)$ is a unique and nonnegative (positive) definite solution to (45). $\square$

Hereafter, we assume that the matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}$, and $\mathcal{Q}$ are constant matrices. Then, the associated algebraic Riccati equation is represented as

$$0 = \mathcal{A}^T \bar{X} + \bar{X}\mathcal{A} - \bar{X}(\mathcal{B}\mathcal{B}^T - \mathcal{D}\mathcal{D}^T)\bar{X} + \mathcal{Q}. \tag{46}$$

We define the notion of a minimal nonnegative definite solution to the algebraic Riccati equation (46) (see also [11]).

Definition 1: A nonnegative definite symmetric matrix $\bar{X}_1$ satisfying the algebraic Riccati equation (46) is said to be *the minimal nonnegative definite symmetric solution* to (46), if given any other nonnegative definite symmetric solution $\bar{X}_2$ to (46), then $\bar{X}_1 \leq \bar{X}_2$.

Lemma 2 follows from [14].

Lemma 2: Suppose that there exists a nonnegative definite

matrix $\bar{X}_1$ satisfying the ARE (46). Then, $X(t, 0)$ exists for all $t \leq T$ and is nondecreasing as t decrease. Moreover, $X(t, 0) \rightarrow \tilde{X} \leq \bar{X}_1$ as $t \rightarrow -\infty$ where $\tilde{X}$ denotes the minimal nonnegative definite solution to the ARE (46).

Lemma 3: Suppose that for $0 \leq X_{T1} \leq X_{T2}$ there exist $X(t, X_{T1})$ and $X(t, X_{T2})$ over $[t_0, T]$. Then, $0 \leq X(t, X_{T1}) \leq X(t, X_{T2})$.

Proof: Consider a value functional

$$I(u, w, x_0) = x(T)^T X_{T2} x(T)$$
$$+ \int_t^T \{x^T Q x + u^T u - w^T w\}\, dt$$

associated with a differential equation

$$\dot{x} = \mathcal{A}x + \mathcal{B}u + \mathcal{D}w, \quad x(0) = x_0.$$

By using the completion of square argument for this value functional, we obtain (see [14])

$$I(u, w, x_0) = x_0^T X(t, X_{T2}) x_0$$
$$+ \int_t^T \{\|u + \mathcal{B}^T X(t, X_{T2})x\|^2$$
$$- \|w - \mathcal{D}^T X(t, X_{T2})x\|^2\}\, dt \tag{47}$$

and

$$I(u, w, x_0) = x_0^T X(t, X_{T1}) x_0$$
$$+ x(T)^T(X_{T2} - X_{T1})x(T)$$
$$+ \int_t^T \{\|u + \mathcal{B}^T X(t, X_{T1})x\|^2$$
$$- \|w - \mathcal{D}^T X(t, X_{T1})x\|^2\}\, dt. \tag{48}$$

Since (47) and (48) hold for any u and w, let $u = -\mathcal{B}^T X(t, X_{T2})x$ and $w = \mathcal{D}^T X(t, X_{T1})x$. Subtracting (48) from (47) yields

$$x_0^T\{X(t, X_{T2}) - X(t, X_{T1})\}x_0$$
$$= x(T)^T(X_{T2} - X_{T1})x(T)$$
$$+ \int_t^T \{\|u + \mathcal{B}^T X(t, X_{T1})x\|^2$$
$$+ \|w - \mathcal{D}^T X(t, X_{T2})x\|^2\}\, dt.$$

This completes the proof since $X_{T2} - X_{T1} \geq 0$ by assumption. $\square$

$X(t, 0)$ is nondecreasing as t decreases and $X(t, 0) \leq X(t, X_T)$ for all $X_T \geq 0$ by Lemma 3. Hence, it is clear that if $X(t, X_T)$ does not exist for $t \leq t_0$, then it increases without bound as t decreases to t_0. On the other hand, if we can find a upper bound for $X(t, X_T)$ for all $t \leq T$, the solution to (44) exists since it can not escape the upper bound.

Corollary 1: Suppose that there exists a nonnegative definite matrix $\bar{X}_1$ satisfying the ARE (46). If $0 \leq X_T \leq \bar{X}_1$ then $X(t, X_T)$ exists and $0 \leq X(t, X_T) \leq \bar{X}_1$ for all $t \leq T$. Moreover, if $0 \leq X_T \leq \tilde{X}$ where $\tilde{X}$ is the minimal nonnegative definite solution to the ARE (46), then

$X(t, X_T) \to \tilde{X}$ as $t \to -\infty$.

Proof: For the first part it suffices that $X(t, X_T)$, if it exists, is bounded above by $\bar{X}_1$. Since $\bar{X}_1 = X(t, \bar{X}_1)$ and $0 \le X_T \le \bar{X}_1$, it follows from Lemma 2 and Lemma 3 that $0 \le X(t, X_T) \le \bar{X}_1$.

From the above discussion and Lemma 3 if $0 \le X_T \le \tilde{X}$, then $X(t, X_T)$ exists and $X(t, 0) \le X(t, X_T) \le \tilde{X}$. Since $X(t, 0) \to \tilde{X}$ as $t \to -\infty$, $X(t, X_T) \to \tilde{X}$ as $t \to -\infty$. $\square$

The solutions developed in Section III-A and III-B, described by (29), (38), (39), (40), and (41), require the solutions of two coupled RDE's (16) and (31). However, these two coupled RDE's can be decoupled into two independent RDE's by a transformation.

The following assumptions result from Section III-A and III-B.

Assumption 1:

a) There exists a solution $P(t)$ to the RDE (16) over $[0, T]$.
b) $P^{-1}(T) + \theta Q_T > 0$.
c) There exists a solution $S(t)$ to the RDE (31) over $[0, T]$.

Note that from Lemma 1, the Assumption 1-a) assures $P(t) > 0$ over $[0, T]$, and the assumptions 1-b) and 1-c) assure $S(t) \ge 0$. Moreover, under the Assumption 1 it is easily verified that $\Pi(t)$ defined in (42) satisfies an RDE

$$-\dot{\Pi} = A^T\Pi + \Pi A - \Pi(BR^{-1}B^T + \theta\Gamma W\Gamma^T)\Pi + Q,$$
$$\Pi(T) = Q_T. \tag{49}$$

Again Lemma 1 shows that $\Pi(t) \ge 0$ over $[0, T]$.

Assumption 2:

a) There exists a solution $P(t)$ to the RDE (16) over $[0, T]$.
b) There exists a solution $\Pi(t)$ to the RDE (49) over $[0, T]$.
c) $P^{-1}(t) + \theta\Pi(t) > 0$ over $[0, T]$.

Claim 1: Assumption 1 and Assumption 2 are equivalent.
Proof: Suppose Assumption 1 holds. Then, $P(t) > 0$ and $\Pi(t) \ge 0$, hence $P^{-1} - \theta S > 0$ over $[0, T]$. By using the matrix inversion lemma

$$P^{-1} + \theta\Pi = P^{-1}(P^{-1} - \theta S)^{-1}P^{-1} > 0.$$

Suppose Assumption 2 holds. Then, $P(T) + \theta\Pi(T) = P(T) + \theta Q_T > 0$. It can be verified that $S = \Pi(I + \theta P\Pi)^{-1}$ satisfies the RDE (31). $\square$

From these relations and by solving the decoupled RDE's (16) and (49) we can construct the controller for each player. For later use we define

$$M = (I + \theta P\Pi)^{-1}P. \tag{50}$$

Then $M(t) > 0$ and satisfies the RDE

$$\dot{M} = M(A - \theta\Gamma W\Gamma^T\Pi)^T + (A - \theta\Gamma W\Gamma^T\Pi)M$$
$$- M(H^T\bar{V}^{-1}H + \theta\Pi BR^{-1}B^T\Pi)M + \Gamma W\Gamma^T \tag{51}$$

with $M(0) = (I + \theta P_0\Pi(0))^{-1}P_0$. It is noted that

$$M = (I - \theta PS)P. \tag{52}$$

D. Saddle Point Strategy

From the results of Section III-B, two strategies for game problem can be deduced from (29), (38), (39), (40), and (41).

Strategy 1:

$$u^* = R^{-1}B^TS\hat{x}, \quad w^* = -\theta W\Gamma^T\Pi x,$$
$$v^* = 0, \quad x^*(0) = \{I + \theta P_0\Pi(0)\}^{-1}\hat{x}_0. \tag{53}$$

Strategy 2:

$$u^* = -R^{-1}B^TS\hat{x}, \quad w^* = -\theta W\Gamma^TS\hat{x},$$
$$v^* = 0, \quad x^*(0) = \{I + \theta P_0\Pi(0)\}^{-1}\hat{x}_0.$$

In the above strategies, x and $\hat{x}$ denote the states of the dynamic equation (1) and the state estimator (15). Note that we have made u^* a function of the estimate in both strategies, but w^* is a function of state in the Strategy 1 and a function of the estimate in the Strategy 2. We will show that the Strategy 1 forms a saddle point under the Assumption 1 while an additional condition is required for the Strategy 2 to produce the saddle point.

By adding the zero quantity

$$0 = -\frac{1}{2}[x^T\Pi x]_0^T + \frac{1}{2}\int_0^T \frac{d}{dt}(x^T\Pi x)\, dt$$

to (6), J is represented as a perfect square of the form

$$J(u, v, w, x(0)) = \frac{1}{2\theta}\| x(0) - \hat{x}_0\|_{P_0^{-1}}^2 + \frac{1}{2}\| x(0)\|_{\Pi(0)}^2$$
$$+ \frac{1}{2}\int_0^T \Big\{\| u + R^{-1}B^T\Pi x\|_R^2$$
$$+ \frac{1}{\theta}(\| w + \theta W\Gamma^T\Pi x\|_{W^{-1}}^2$$
$$+ \| v\|_{\bar{V}^{-1}}^2)\Big\}\, dt. \tag{54}$$

Define $\xi \triangleq x - (I - \theta PS)\hat{x}$. Then, ξ satisfies

$$\dot{\xi} = (A - MH^T\bar{V}^{-1}H)\xi + \theta PSB\hat{u}$$
$$+ \Gamma\hat{w} - MH^T\bar{V}^{-1}\Gamma_1 v \tag{55}$$

with the initial condition $\xi(0) = x(0) - \{I - \theta P_0S(0)\}\hat{x}_0$ where $\hat{u} = u + R^{-1}B^TS\hat{x}$ and $\hat{w} = w + \theta W\Gamma^TS\hat{x}$. Adding

$$0 = -\frac{1}{2}\Big[\| \hat{x}\|_S^2 - \frac{1}{\theta}\| x - \hat{x}\|_{P^{-1}}^2\Big]_0^T$$
$$+ \frac{1}{2}\int_0^T \frac{d}{dt}\Big[\| \hat{x}\|_S^2 - \frac{1}{\theta}\| x - \hat{x}\|_{P^{-1}}^2\Big]\, dt$$

to (6), yields J as

$$J(u, v, w, x(0)) = \frac{1}{2}\| \hat{x}_0\|_{S(0)}^2 + \frac{1}{2\theta}\| \xi(T)\|_{M^{-1}(T)}^2$$

$$+ \frac{1}{2} \int_0^T \left\{ \| \hat{u} \|_R^2 \right.$$

$$+ \frac{1}{\theta} (\| \hat{w} - W \Gamma^T P^{-1} \xi \|_{W^{-1}}^2$$

$$+ \left. \| \Gamma_1 v + H \xi \|_{\bar{V}^{-1}}^2) \right\} \, dt. \qquad (56)$$

Note that (54), (55), and (56) hold for any strategy. For convenience let u_1^*, w_1^*, v_1^*, and $x_1^*(0)$ denote Strategy 1 and Strategy 2 is indexed by the subscript 2.

Proposition 1: Under the Assumption 1, Strategy 1 forms a saddle point, that is,

$$J(u_1^*, v, w, x(0)) \le J(u_1^*, v_1^*, w_1^*, x_1^*(0))$$
$$\le J(u, v_1^*, w_1^*, x_1^*(0)) \qquad (57)$$

for all $u, v, w \in L_2[0, T]$, $x(0) \in R^n$.

Proof: If $u = u_1^*$, then $\hat{u} = 0$. Hence from (56)

$$J(u_1^*, v, w, x(0)) = \frac{1}{2} \| \hat{x}_0 \|_{S(0)}^2 + \frac{1}{2\theta} \| \xi(T) \|_{M^{-1}(T)}^2$$

$$+ \frac{1}{2\theta} \int_0^T (\| \hat{w} - W \Gamma^T P^{-1} \xi \|_{W^{-1}}^2$$

$$+ \| \Gamma_1 v + H \xi \|_{\bar{V}^{-1}}^2) \, dt.$$

Since $M^{-1}(T) > 0$ under the Assumption 1, it is obvious that

$$J(u_1^*, v, w, x(0)) \le \frac{1}{2} \| \hat{x}_0 \|_{S(0)}^2. \qquad (58)$$

From (43) and (58), we obtain the left-hand side inequality in (57).

From (54) we obtain

$$J(u, v_1^*, w_1^*, x_1^*(0)) = \frac{1}{2} \left[\| \hat{x}_0 \|_{S_0}^2 \right.$$

$$+ \left. \int_0^T \| u + R^{-1} B^T \Pi x \|_R^2 \, dt \right].$$

Therefore

$$J(u_1^*, v_1^*, w_1^*, x_1^*(0)) = \frac{1}{2} \| \hat{x}_0 \|_{S(0)}^2 \le J(u, v_1^*, w_1^*, x_1^*(0))$$

which completes the proof. □

Proposition 2: Under the Assumption 1

$$J(u_2^*, v, w, x(0)) \le J(u_2^*, v_2^*, w_2^*, x_2^*(0)) \qquad (59)$$

for all $v, w \in L_2[0, T]$, $x(0) \in R^n$. In addition, if there exists a real symmetric solution to the RDE

$$- \dot{\mathcal{F}} = (A - MH^T \bar{V}^{-1} H)^T \mathcal{F} + \mathcal{F}(A - MH^T \bar{V}^{-1} H)$$

$$- \theta \mathcal{F} PSBR^{-1} B^T SP \mathcal{F} + P^{-1} \Gamma W \Gamma^T P^{-1} + H^T \bar{V}^{-1} H \qquad (60)$$

with the final condition $\mathcal{F}(T) = M^{-1}(T)$, then

$$J(u_2^*, v_2^*, w_2^*, x_2^*(0)) \le J(u, v_2^*, w_2^*, x_2^*(0)),$$

$$\forall u \in L_2[0, T]. \qquad (61)$$

Proof: The proof of the inequality (59) is the same as that of the left-hand side inequality in (57).

Let ξ^* denote the output of (55) when w_2^*, v_2^*, and $x_2^*(0)$ are applied to the system dynamics and the state estimator. Then, from (55) ξ^* satisfies

$$\dot{\xi}^* = (A - MH^T \bar{V}^{-1} H) \xi^* + \theta PSB \hat{u}, \quad \xi^*(0) = 0$$

and

$$J(u, v_2^*, w_2^*, x_2^*(0)) = \frac{1}{2} \| \hat{x}_0 \|_{S(0)}^2 + \frac{1}{2\theta} \| \xi^*(T) \|_{M^{-1}(T)}^2$$

$$+ \frac{1}{2} \int_0^T \left\{ \| \hat{u} \|_R^2 \right.$$

$$+ \frac{1}{\theta} (\| W \Gamma^T P^{-1} \xi^* \|_{W^{-1}}^2$$

$$+ \left. \| H \xi^* \|_{\bar{V}^{-1}}^2) \right\} \, dt. \qquad (62)$$

If there exists a real symmetric solution $\mathcal{F}$ to the RDE (60) over $[0, T]$, then adding

$$0 = - \frac{1}{2\theta} \left[\xi^{*T} \mathcal{F} \xi^* \right]_0^T + \frac{1}{2\theta} \int_0^T \frac{d}{dt} (\xi^{*T} \mathcal{F} \xi^*) \, dt$$

to (62) yields J as a perfect square of the form

$$J(u, v_2^*, w_2^*, x_2^*(0)) = \frac{1}{2} \| \hat{x}_0 \|_{S(0)}^2$$

$$+ \frac{1}{2} \int_0^T \left\| \hat{u} + \frac{1}{\theta} R^{-1} B^T SP \mathcal{F} \xi^* \right\|_R^2 \, dt. \qquad (63)$$

Consequently comparing (43) and (63) yields the inequality (61). □

Suppose that u, v, w, and $x(0)$ play with the Strategy 2. By taking the coordinate transformation $x_c = (I + \theta P \Pi)^{-1} \hat{x}$, we obtain the following strategy, say Strategy 3.

Strategy 3:

$$u^* = - R^{-1} B^T \Pi x_c$$

$$w^* = - \theta W \Gamma^T \Pi x_c, \quad v^* = 0,$$

$$x^*(0) = \{ I + \theta P_0 \Pi(0) \}^{-1} \hat{x}_0 \qquad (64)$$

where

$$\dot{x}_c = A x_c + Bu - \theta \Gamma W \Gamma^T \Pi x_c + MH^T \bar{V}^{-1}(z - H x_c) \qquad (65)$$

with $x_c(0) = (I + \theta P_0 \Pi(0))^{-1} \hat{x}_0$. Since Strategy 3 is obtained by a coordinate transformation of Strategy 2, it can be assumed that this strategy is equivalent to Strategy 2. However, Proposition 3 shows that this conjecture is false.

Proposition 3: With Assumption 1, Strategy 3 satisfies the saddle point condition

$$J(u_3^*, v, w, x(0)) \le J(u_3^*, v_3^*, w_3^*, x_3^*(0))$$

$$\le J(u, v_3^*, w_3^*, x_3^*(0)) \qquad (66)$$

for all $u, v, w \in L_2[0, T]$, $x(0) \in R^n$.

Proof: First, prove the left side inequality. If u plays st with the Strategy 3, then it can be verified that

$$x_c = (I + \theta P\Pi)^{-1}\hat{x} \qquad (67)$$

r any v, w, and $x(0)$. Therefore, $u_3^* = u_1^*$ since $S = \Pi(I \cdot \theta P\Pi)^{-1}$.

Next, the right side inequality of (66) is proved. Suppose v, and $x(0)$ play first with Strategy 3. If u does not ay its saddle point strategy, then (67) does not hold. owever, comparing (1) to (65), we obtain that $x = x_c$ for iy u, hence $w_3^* = w_1^*$. From this consideration, Strategy 3 equivalent to Strategy 1. $\qquad \square$

In controller (15) and (53), the worst case disturbances w^* ad $x^*(0)$ are not explicit as they are in controller (64) and 55). Clearly, if w, v, and $x(0)$ play Strategy 3, then by bserving (65), the error $x - x_c = 0$ even if u does not play ptimally. In contrast, the error ζ^*, defined as $\zeta^* = x - (I + \theta P\Pi)^{-1}\hat{x}$ and propagated by (55) where w, v, and $x(0)$ lay Strategy 2, is nonzero if $\hat{u} \neq 0$. This is because (67) nly holds when u plays its saddle point strategy.

If u is confined to be a linear function of Z_t, then the ollowing lemma holds.

Lemma 4: There exists a $\bar{u} \in \mathcal{U}_l$ such that $J(\bar{u}, v, w, \cdot(0))$ is strictly concave with respect to $(v, w, x(0))$ if and nly if Assumption 1 or, equivalently, Assumption 2 holds. Moreover, when the assumption holds, $\bar{u} = -R^{-1}B^T S\hat{x}$ is a ontrol which makes $J(\bar{u}, v, w, x(0))$ strictly concave with espect to $(v, w, x(0))$.

Proof: See appendix. $\qquad \square$

Since the transformation (67) is valid for any $v, w, x(0)$ or $\bar{u} = -R^{-1}B^T S\hat{x}$, $\bar{u} = -R^{-1}B^T \Pi x_c$ is equivalent to $\bar{u} = -R^{-1}B^T S\hat{x}$.

As seen in Propositions 1 and 2, for a given $\bar{u} \in \mathcal{U}_l$ which makes $J(\bar{u}, v, w, x(0))$ strictly concave with repect to $(v, w, x(0))$ we can find many optimal strategies $(v^*, w^*, x^*(0))$ of $(v, w, x(0))$ which satisfies

$$J(\bar{u}, v, w, x(0)) \leq J(\bar{u}, v^*, w^*, x^*(0)),$$
$$\forall v, w \in L_2[0, T], \; x(0) \in R^n.$$

However, all optimal strategies produce the same optimal rajectory, that is, the optimal trafectory is unique. Therefore

$$J(\bar{u}, v, w, x(0)) < J(\bar{u}, v^*, w^*, x^*(0))$$

or all $v, w \in L_2[0, T]$, $x(0) \in R^n$ such that $(v(t), w(t), c(0)) \neq (v^*(t), w^*(t), x^*(0))$ for all $t \in [0, T]$.

E. The Deterministic Linear-Quadratic Game and the LEG Stochastic Control Problem

The solution of the deterministic linear-quadratic game problem considered here, in particular (15), (16), (29), and 31), is equivalent to the solution of the linear-exponential-Jaussian problem [16]

$$\min_u E\left[-\theta \exp\left\{-\theta\left(\int_0^T (x^T Q x + u^T R u)\, dt \right.\right.\right.$$
$$\left.\left.\left. + x(T)^T Q_T x(T)\right)\right\}\right]$$

subject to the stochastic linear system

$$\dot{x} = Ax + Bu + \Gamma w,$$
$$z = Hx + \Gamma_1 v$$

where $E[\cdot]$ denotes the expectation operator, $x(0)$ is normally distributed with mean $\hat{x}_0$ and covariance P_0, and $w(t)$ and $v(t)$ are jointly Gaussian independent white noise processes with statistic

$$E[w(t)] = 0, \quad E[w(t)w(\tau)^T] = W(t)\delta(t - \tau),$$
$$E[v(t)] = 0, \quad E[v(t)v(\tau)^T] = V(t)\delta(t - \tau)$$

where $\delta(t - \tau)$ denotes the Dirac delta function. It should be noted that the conditions for optimality determined here are not the same as the conditions that are required for the solution in [16]. However, the conditions in [17], [18] do reduce to those given here.

IV. FINITE-TIME INTERVAL DISTURBANCE ATTENUATION PROBLEM

In this section, the finite-time interval disturbance attenuation problem is solved by using the results in Section III.

Let J_0 denote J with $\hat{x}_0 = 0$. The finite-time disturbance attenuation problem (4) is equivalent to finding $u \in \mathcal{U}_l$ satisfying

$$J_0(u, v, w, x(0)) < 0$$

for all v, $w \in L_2[0, T]$, $x(0) \in R^n$, such that $(v(t), w(t), x(0)) \neq 0$ over all $t \in [0, T]$.

Theorem 1: There exists a solution $u \in \mathcal{U}_l$ to the finite-time disturbance attenuation problem (4) if and only if Assumption 1 or 2 holds. If the assumption holds, $u = -R^{-1}B^T S\hat{x} = -R^{-1}B^T \Pi x_c$ is a solution.

Proof:

Sufficiency: Suppose that assumption 1 or 2 holds. Then, for $u = u_1^* = -R^{-1}B^T S\hat{x}$, from Proposition 1

$$J_0(u_1^*, v, w, x(0)) \leq J_0(u_1^*, v_1^*, w_1^*, x_1^*(0)) = 0$$

for all v, $w \in L_2[0, T]$, $x(0) \in R^n$. Since $\hat{x}_0 = 0$, $x_1^*(0) = 0$. Hence, for u_1^*, v_1^*, w_1^*, and $x_1^*(0)$, equations (1) and (15) become homogeneous equations with zero initial conditions, which leads $x(t) = \hat{x}(t) = 0$ over $[0, T]$ and $(v_1^*(t), w_1^*(t), x_1^*(0)) = 0$ over $[0, T]$. From Lemma 4, $J_0(u_1^*, v, w, x(0))$ is strictly concave with respect to $(v, w, x(0))$, hence

$$J_0(u_1^*, v, w, x(0)) < 0$$

for all v, $w \in L_2[0, T]$, $x(0) \in R^n$ such that $(v(t), w(t), x(0)) \neq 0$ for all $t \in [0, T]$.

Necessity: Suppose that $\bar{u} \in \mathcal{U}_l$ is a solution to (4). Since $\bar{u} \in \mathcal{U}_l$, $J_0(\bar{u}, v, w, x(0))$ is a quadratic function with respect to $(v, w, x(0))$. For $(v(t), w(t), x(0)) = 0$ over $[0, T]$, $z(t) = Hx(t)$ from which the equation (1) with $u = \bar{u}$ becomes a homogeneous equation with zero initial condition, and $\bar{u}(t) = 0$ over $[0, T]$. Therefore,

$$J_0(\bar{u}, v, w, x(0)) \leq J_0(\bar{u}, 0, 0, 0) = 0$$
$$\forall v, w \in L_2[0, T], \; x(0) \in R^n$$

and $(v(t), w(t), x(0)) = 0$, $t \in [0, T]$ is a unique extremal

trajectory, that is, $J_0(\bar{u}, v, w, x(0))$ is strictly concave with respect to $(v, w, x(0))$. Applying Lemma 4 completes the proof. $\qquad\square$

V. TIME-INVARIANT CONTROLLER

In this section, we assume that the systems (1), (2), and (3) are time-invariant systems with zero initial condition; that is, A, B, Γ, H, Γ_1, C, and C_1 are constant matrices. It is also assumed that all weighting matrices in (6) are constant, in particular, W and V are identity matrices. It is also assumed that (A, B) and (A, Γ) are stabilizable and controllable pairs, respectively, and (H, A) and (C, A) are detectable pairs. A disturbance attenuation problem for this system has been solved in [1] based on two ARE's associated with the RDE's of (16) and (49). In this section, all stabilizing compensators which can be constructed from the solutions of two ARE's satisfy an H_∞ norm bound.

Suppose that there exists a nonnegative definite $\bar{\Pi}$ and a positive definite $\bar{P}$ satisfying the following ARE's

$$0 = A^T\bar{\Pi} + \bar{\Pi}A - \bar{\Pi}(BR^{-1}B^T + \theta\Gamma\Gamma^T)\bar{\Pi} + C^TC \quad (68)$$

$$0 = A\bar{P} + \bar{P}A^T - \bar{P}(H^T\bar{V}^{-1}H + \theta C^TC)\bar{P} + \Gamma\Gamma^T \quad (69)$$

such that

$$\bar{P}^{-1} + \theta\bar{\Pi} > 0. \quad (70)$$

By Corollary 1 the solution $\Pi(t)$ to the RDE (49) converges to $\bar{\Pi}_m$, if $Q_T \leq \bar{\Pi}_m$ where $\bar{\Pi}_m$ is the minimal nonnegative definite solution to the ARE (68). The solution to RDE (16) has similar properties.

As $T \to \infty$, the compensator, described by (64) and (65) becomes a time-invariant controller of the form

$$\dot{x}_c = A_c x_c + B_c z$$
$$u = C_c x_c \quad (71)$$

where

$$A_c = A - BR^{-1}B^T\bar{\Pi} - \bar{M}H^T\bar{V}^{-1}H - \theta\Gamma\Gamma^T\bar{\Pi}$$

$$B_c = \bar{M}H^T\bar{V}^{-1}$$

$$C_c = -R^{-1}B^T\bar{\Pi}$$

$$\bar{M} = (I + \theta\bar{P}\bar{\Pi})^{-1}\bar{P} > 0.$$

Note that we can also obtain a time-invariant controller from (15) and (53) which can be transformed to (71) by taking a coordinate transformation $x_c = (I + \theta\bar{P}\bar{\Pi})^{-1}\hat{x}$. In general, there can be more than one nonnegative definite solution to (68) or (69). Therefore, we can construct more than one time-invariant controller from (68) and (69). It is noted that $\bar{M}$ is a positive definite solution to the ARE resulting from the RDE (51)

$$0 = \bar{M}(A - \theta\Gamma\Gamma^T\bar{\Pi})^T + (A - \theta\Gamma\Gamma^T\bar{\Pi})\bar{M}$$
$$- \bar{M}(H^T\bar{V}^{-1}H + \theta\bar{\Pi}BR^{-1}B^T\bar{\Pi})\bar{M} + \Gamma\Gamma^T. \quad (72)$$

By using the time-invariant controller (71) the closed-loop system becomes

$$\dot{\bar{x}} = A_{cl}\bar{x} + \Gamma_{cl}\bar{w}$$
$$y = C_{cl}\bar{x} \quad (73)$$

where

$$\bar{x} = \begin{bmatrix} x \\ x_c \end{bmatrix}, \quad \bar{w} = \begin{bmatrix} w \\ v \end{bmatrix}, \quad A_{cl} = \begin{bmatrix} A & BC_c \\ B_cH & A_c \end{bmatrix},$$

$$\Gamma_{cl} = \begin{bmatrix} \Gamma & 0 \\ 0 & B_c\Gamma_1 \end{bmatrix}, \quad C_{cl} = \begin{bmatrix} C & 0 \\ 0 & C_1C_c \end{bmatrix}.$$

The transfer function from the disturbance $\bar{w}$ to y denoted by T_{yw} is given as

$$T_{yw}(s) = C_{cl}[sI - A_{cl}]^{-1}\Gamma_{cl}.$$

Preposition 4: The closed-loop system (73) is stable and

$$\|T_{yw}\|_\infty \leq \frac{1}{\sqrt{-\theta}}. \quad (74)$$

Claim 2: $A - \bar{M}H^T\bar{V}^{-1}H - \theta\Gamma\Gamma^T\bar{\Pi}$ is a stable matrix.
  Proof: Rewrite (72) as

$$\bar{M}(A - \bar{M}H^T\bar{V}^{-1}H - \theta\Gamma\Gamma^T\bar{\Pi})^T$$
$$+ (A - \bar{M}H^T\bar{V}^{-1}H - \theta\Gamma\Gamma^T\bar{\Pi})\bar{M}$$
$$= -\{\bar{M}(H^T\bar{V}^{-1}H - \theta\bar{\Pi}BR^{-1}B^T\bar{\Pi})\bar{M} + \Gamma\Gamma^T\}$$
$$\triangleq -U.$$

Since (A, Γ) is controllable, $[A - \theta\Gamma\Gamma^T\bar{\Pi}, \Gamma]$ is controllable. From [22, Lemma 4.1], $[A - \bar{M}H^T\bar{V}^{-1}H - \theta\Gamma\Gamma^T\bar{\Pi}, (\bar{M}H^T\bar{V}^{-1}H\bar{M} + \Gamma\Gamma^T)^{1/2}]$ is controllable. By applying the lemma again $[A - \bar{M}H^T\bar{V}^{-1}H - \theta\Gamma\Gamma^T\bar{\Pi}, U^{1/2}]$ is controllable. Since $\bar{M} > 0$ is the solution of above Lyapunov equation, the claim is completed by using [22, Lemma 4.2]. $\qquad\square$

  Proof of Proposition 4: It can be verified that $\mathcal{X}$ defined as

$$\mathcal{X} = \begin{bmatrix} \bar{P}^{-1} & -\bar{M}^{-1} \\ -\bar{M}^{-1} & \bar{M}^{-1} \end{bmatrix}$$

satisfies the ARE

$$A_{cl}^T\mathcal{X} + \mathcal{X}A_{cl} = -\mathcal{X}\Gamma_{cl}\Gamma_{cl}^T\mathcal{X} + \theta C_{cl}^TC_{cl}. \quad (75)$$

Since (C, A) is detectable, there exists a L such that $A - LC$ is stable. For this L

$$A_{cl} - \begin{bmatrix} L & BR^{-1}C_1^T \\ 0 & BR^{-1}C_1^T \end{bmatrix}C_{cl}$$

$$= \begin{bmatrix} A - LC & 0 \\ B_cH & A - \bar{M}H^T\bar{V}^{-1}H - \theta\Gamma\Gamma^T\bar{\Pi} \end{bmatrix}$$

which from Claim 2 implies that (C_{cl}, A_{cl}) is detectable. [22, Lemma 4.1] shows that $[(\mathcal{X}\Gamma_{cl}\Gamma_{cl}^T\mathcal{X} - \theta C_{cl}^TC_{cl})^{1/2}, A_{cl}]$ is dectectable. Observe that $\bar{P}^{-1} = \bar{M}^{-1} - \theta\bar{\Pi}$, hence

$$\mathcal{X} = \begin{bmatrix} \bar{M}^{-1/2} \\ -\bar{M}^{-1/2} \end{bmatrix}\begin{bmatrix} \bar{M}^{-1/2} \\ -\bar{M}^{-1/2} \end{bmatrix}^T - \theta\begin{bmatrix} \bar{\Pi} & 0 \\ 0 & 0 \end{bmatrix} \geq 0.$$

Therefore, it follows from [22, Lemma 4.2] that A_{cl} is stable.

The H_∞ bound can be deduced from the inequality (57) or (66) as in [7] which considered the state feedback case.

nsider the case where $\hat{x}_0 = 0$, $P_0 = \bar{P}$, $Q_T = \bar{\Pi}$. Then control law (71) is equivalent to the control law for u in ategy 3. When the controller (71) is used, from the t-hand side inequality of (66)

$$J(u_3^*, v, w, 0) \le 0, \qquad \forall w, v \in L_2[0, T]. \qquad (76)$$

reover, since the closed-loop (73) is stable, as T goes to inity

$$t), x_c(t), u(t), y(t) \in L_2[0, \infty) \qquad \forall w, v \in L_2[0, \infty)$$

m which $x(\infty) = 0 \ \forall w, v \in L_2[0, \infty)$. ...nce, from (76) obtain

$$y^T y \, dt \le -\frac{1}{\theta} \int_0^\infty (w^T w + v^T v) \, dt$$

$$\forall w, v \in L_2[0, \infty)$$

ich is equivalent to (74). □

Note that for all solutions $\bar{\Pi} \ge 0$ and $\bar{P} > 0$ to (68) and 9), respectively, such that $\bar{P}^{-1} + \theta \bar{\Pi} > 0$, Proposition 4 lds. If we take the minimal nonnegative and positive finite solutions as $\bar{\Pi}$ and $\bar{P}$, then the time-invariant con-ller (71) is equivalent to the H_∞ controller proposed in].

Example: Consider evaluating the controller (71) for the alar system

$$\dot{x} = -1.5x + u + w$$

$$z = x + \frac{1}{\sqrt{14}} v$$

$$Q = 4, \quad R = 2, \quad \theta = -1.$$

e corresponding ARE's are

$$-3\bar{\Pi} + 0.5\bar{\Pi}^2 + 4 = 0 \Rightarrow \bar{\Pi} = 2, 4$$

$$-3\bar{P} - 10\bar{P}^2 + 1 = 0 \Rightarrow \bar{P} = 0.2.$$

e obtain two positive solutions for $\bar{\Pi}$ which satisfy the equality (48). Therefore, two controllers can be con-ructed.

For $(\bar{\Pi}, \bar{P}) = (2, 0.2)$

$$\left. \begin{array}{l} \dot{x}_c = -5.167x + 4.667z \\ u = -x_c \end{array} \right\} \quad (*)$$

For $(\bar{\Pi}, \bar{P}) = (4, 0.2)$

$$\left. \begin{array}{l} \dot{x}_c = -13.5x + 14z \\ u = -2x_c \end{array} \right\} \quad (**).$$

g. 1 depicts the largest singular value of the $T_{yw}(j\omega)$ of the sed-loop system using the controller (*) or (**), and ows that the two controllers satisfy the H_∞-norm bound of ,w. In this example as θ increases in a negative way, we il to obtain a positive solution to the $\bar{\Pi}$ equation for < -17116. This occurs when the Hamiltonian matrix as-ciated with the $\bar{\Pi}$ equation has two eigenvalues at the igin.

VI. CONCLUSIONS

A finite-time disturbance attenuation problem was analyzed a game theoretic approach. By adopting a calculus of
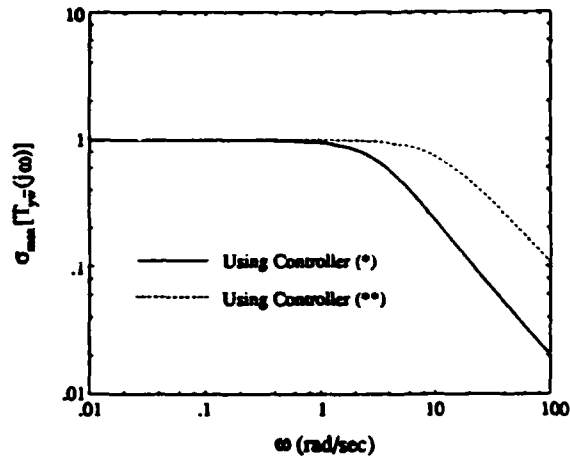

Fig. 1. Largest singular value plot.

variation technique to solve a linear-quadratic differential game, an extremely straightforward derivation is obtained for a linear n-dimensional compensator. A solution to a finite-time disturbance attenuation problem was obtained by using the results of a LQ game problem with partial information. The resulting compensator can be time-varying, thereby gen-eralizing the results of [1]. The generality of the problem formulation and the simplicity of its solution is due to the use of a state space rather than a frequency domain approach. In Section III-D it is shown that this approach suggests that there could be more than one saddle strategy. Three strate-gies were considered and sufficient conditions are given for when they satisfy the saddle point condition. Furthermore, it is shown that there exists a $\bar{u} \in \mathcal{U}_I$ which makes $J(\bar{u}, w, v, x(0))$ strictly concave if and only if Assumption 1 or 2 is satisfied. Also, the controller u^* given in Section III-C is identical to that of the LEG problem [16] except that certain conditions found in [16] are different.

By specializing to time-invariant systems and infinite-time cost criterion, a time-invariant controller results. To do this we studied the properties of the two RDE's obtained in Section III-A and III-B, and decomposed their associated ARE's in Section III-C. It was shown that there can be more than one nonnegative definite solution to the ARE and that the H_∞ norm given by (74) is satisfied by all compensators constructed from all of them which satisfy certain conditions. It is suggested that the minimal nonnegative definite solution to the ARE which has a strong solution [11] be used to design the compensator. Finally, we gave a direct proof that the LEG and differential game controllers satisfy the H_∞ bound [3].

APPENDIX

For a given $\bar{u} \in \mathcal{U}_I$, let $\bar{J}(v, w, x(0)) = J(\bar{u}, v, w, x(0))$. The second variation $\delta^2 \bar{J}$ along the extremal path is given as

$$\delta^2 \bar{J} = \frac{1}{2} \left[\frac{1}{\theta} \| \delta x(0) \|^2_{P_0^{-1}} + \| \delta x(T) \|^2_{Q_T} \right.$$

$$\left. + \int_0^T \left\{ \| \delta x \|^2_Q + \| \delta \bar{u} \|^2_R \right. \right.$$

$$+ \frac{1}{\theta} \left(\| \delta w \|^2_{W^{-1}} + \| \delta v \|^2_{V^{-1}} \right) \Big\} \, dt \Bigg].$$

Taking a first variation of (1), (2), and (15) for given $u = \bar{u}$, we obtain

$$\delta \dot{x} = A \, \delta x + B \, \delta \bar{u} + \Gamma \, \delta w, \qquad (77)$$

$$\delta z = H \, \delta x + \Gamma_1 \, \delta v \qquad (78)$$

$$\delta \dot{\hat{x}} = A \, \delta \hat{x} + B \, \delta \bar{u} + PH^T \bar{V}^{-1} (\delta z - H \, \delta \hat{x}) - \theta PQ \, \delta \hat{x},$$

$$\delta \hat{x}(0) = 0 \qquad (79)$$

Note that (79) is defined over the interval where $P(t)$ exists. Define $e(t) = \hat{x}(t) - x(t)$. Then, δe satisfies

$$\delta \dot{e} = (A - PH^T \bar{V}^{-1} H) \, \delta e - \Gamma \, \delta w + PH^T \bar{V}^{-1} \Gamma_1 \, \delta v$$
$$\qquad - \theta PQ \, \delta \hat{x} \qquad (80)$$

with a initial condition $e(0) = - \delta x(0)$.

Claim 3: Suppose that $P(t)$ exists over $[0, t_1]$ and $\bar{u} \in \mathcal{U}_t$ is given. If $\delta v = \Gamma_1^{-1} H \, \delta e$ over $[0, t_1]$, then $\delta \hat{x}(t) = 0$ and $\delta \bar{u}(t) = 0$ over $[0, t_1]$. In addition, if δw is linear with respect to δe, then any nonzero $\delta e(t_2)$, $t_2 \in [0, t_1]$, can be produced with an appropriate choice of nonzero $\delta x(0)$.

Proof: If $\delta v = \Gamma_1^{-1} H \, \delta e$, then $\delta z = H \, \delta x + \Gamma_1 \, \delta v = H \, \delta \hat{x}$. Equation (79) for given $u = \bar{u}$, therefore, becomes

$$\delta \dot{\hat{x}} = A \, \delta \hat{x} + B \, \delta \bar{u} - \theta PQ \, \delta \hat{x}, \quad \delta \hat{x}(0) = 0$$

over $[0, t_1]$. Since δz is a function of only $\delta \hat{x}$ and $\bar{u}$ is linear with respect to Z_t, $\delta \hat{x}(t) = 0$ and $\delta \bar{u}(t) = 0$ over $[0, t_1]$. Equation (80) becomes

$$\delta \dot{e} = A \, \delta e - \Gamma \, \delta w, \quad \delta e(0) = - \delta x(0), \quad t \in [0, t_1].$$

If δw is linear in δe, then

$$\delta e(t) = - \Phi_1(t, t_0) \, \delta x(0)$$

where Φ_1 is the transition matrix of resulting linear system. This complete the proof. $\qquad \square$

Proof of Lemma 4:

Sufficiency: Suppose the Assumption 1 holds. Consider $\bar{u} = -R^{-1} B^T S \hat{x}$. Then $\delta \bar{u} = -R^{-1} B^T S \, \delta \hat{x}$. In a similar way in (56), we obtain

$$\delta^2 \bar{J} = \frac{1}{2\theta} \Bigg[\| \delta \xi(T) \|^2_{M^{-1}(T)} + \int_0^T \Big\{ \| \delta \hat{w} - W \Gamma^T$$
$$\cdot P^{-1} \delta \xi \|^2_{W^{-1}} + \| \Gamma_1 \delta v + H \delta \xi \|^2_{V^{-1}} \Big\} \Bigg] dt$$

where

$$\delta \dot{\xi} = (A - MH^T \bar{V}^{-1} H) \, \delta \xi + \Gamma \, \delta \hat{w} - MH^T \bar{V}^{-1} \Gamma_1 \, \delta v,$$

$$\delta \xi(0) = \delta x(0).$$

The equality holds only for $\delta \xi(T) = 0$, $\delta \hat{w} = W \Gamma^T P^{-1} \delta \xi$ and $\Gamma_1 \delta v = - H \delta \xi$. For these values

$$\delta \dot{\xi} = (A + \Gamma W \Gamma^T P^{-1}) \, \delta \xi, \quad \delta \xi(T) = 0$$

which implies $\delta \xi(t) = 0$ over $[0, T]$, hence $\delta v(t) = 0$, $\delta w(t) = - \theta W \Gamma^T S \, \delta \hat{x}(t)$ over $[0, T]$ and $\delta x(0) = 0$. From (77) and (78), $\delta \hat{x}(t) = 0$. Therefore, $\delta^2 \bar{J} = 0$ only if

$(\delta v(t), \delta w(t), \delta x(0)) = 0$ over all $t \in [0, T]$, from which

$$\delta^2 \bar{J} > 0$$

for all $\delta v, \delta w, \delta x(0)$ such that $(\delta v(t), \delta w(t), \delta x(0)) \neq 0$ for all $t \in [0, T]$, that is, $\bar{J}$ is strictly concave with respect to $(v, w, x(0))$.

Necessity: Suppose that for a $\bar{u} \in \mathcal{U}_t$, $\bar{J}$ is strictly concave with respect to $(v, w, x(0))$. Since J is a quadratic function, the second variation of J along the extremal path $\delta^2 \bar{J}$ should be negative for all $\delta v, \delta w, \delta x(0) \neq 0$ such that $(\delta v(t), \delta w(t), \delta x(0)) \neq 0$ for all $t \in [0, T]$.

a) Suppose assumption (2a) is violated. Let $t_1 \in (0, T]$ be the escape time to RDE (16). Then, for some nonzero vector ρ_1, $\rho_1^T P^{-1}(t) \rho_1 \to 0$ as $t(< t_e) \to t_e$. Adding

$$0 = \frac{1}{2} \left[\delta e^T (\theta P)^{-1} \delta e \right]_0^{t_s} - \frac{1}{2} \int_0^{t_s} \frac{d}{dt} \left\{ \delta e^T (\theta P)^{-1} \delta e \right\} dt$$

where $0 \leq t_s < t_1$ to $\delta^2 \bar{J}$ yields

$$2 \delta^2 \bar{J} = \int_0^{t_s} \Big\{ \| \delta \hat{x} \|^2_Q + \| \delta \bar{u} \|^2_R$$
$$+ \| \delta w + \theta W \Gamma^T P^{-1} \delta e \|^2_{(\theta W)^{-1}}$$
$$+ \| \Gamma_1 \delta v - H \delta e \|^2_{(\theta \bar{V})^{-1}} \Big\} dt + \| \delta e(t_s) \|^2_{(\theta P(t_s))^{-1}}$$
$$+ \| \delta x^T(T) \|^2_{Q_T} + I(t_s : T)$$

where

$$I(t_s : T) = \int_{t_s}^T \Big\{ \| \delta x \|^2_Q + \| \delta \bar{u} \|^2_R + \| \delta w \|^2_{(\theta W)^{-1}}$$
$$+ \| \delta v \|^2_{(\theta V)^{-1}} \Big\} dt.$$

Choose

$$\delta w = \theta W \Gamma^T P^{-1} \delta e, \quad \delta v = - \Gamma_1^{-1} H \delta e; \quad \text{for } 0 \leq t < t_s$$
$$\delta w = 0, \quad \delta v = 0; \quad \text{for } t_s \leq t \leq T.$$

From Claim 3, $\delta \hat{x}(t) = 0$ and $\delta \bar{u}(t) = 0$ over $[0, t_s]$, and $\delta x(0) \neq 0$ can be chosen as $\delta e(t_s) = \rho_1$. Hence

$$2 \delta^2 \bar{J} = \| \rho_1 \|^2_{(\theta P(t_s))^{-1}} + \| \delta x^T(T) \|^2_{Q_T}$$
$$+ \int_{t_s}^T \left(\| \delta x \|^2_Q + \| \delta \bar{u} \|^2_R \right) dt.$$

As $t_s \to t_1$, $\delta^2 \bar{J} \geq 0$ which is a contradiction.

b) Suppose that assumption 2a) holds, but Assumption 2b) is violated. Let $t_2 \in [0, T)$ be the escape time to RDE (49). Then for some nonzero vector ρ_2, $\rho_2^T \Pi(t) \rho_2 \to \infty$ as $t(> t_e) \to t_e$. Adding

$$0 = + \frac{1}{2} \left[\delta x^T \Pi \delta x \right]_{t_s}^T - \frac{1}{2} \int_{t_s}^T \frac{d}{dt} \left\{ \delta x^T \Pi \delta x \right\} dt$$

where $t_2 \leq t_s \leq T$ to $\delta^2 J$ yields

$$2 \delta^2 \bar{J} = \| \delta x(0) \|^2_{(\theta P_0)^{-1}} + \| \delta x(t_s) \|^2_{\Pi(t_s)} + I(0 : t_s)$$
$$+ \int_{t_s}^T \Big\{ \| \delta \bar{u} + R^{-1} B^T \Pi \delta x \|^2_R$$

$$+ \| \delta w + \theta W \Gamma^T \Pi \, \delta x \|^2_{(\theta W)^{-1}} + \| \delta v \|^2_{(\theta V)^{-1}} \} \, dt.$$

Choose

$$\delta w = 0, \quad \delta v = \Gamma_1^{-1} H \delta e; \qquad \text{for } 0 \le t \le t_s$$

$$\delta w = -\theta W \Gamma^T \Pi \, \delta x, \quad \delta v = 0; \qquad \text{for } t_s < t \le T$$

and $\delta x(0) \neq 0$ as $\delta x(t_s) = \rho_2$. Hence

$$2 \delta^2 \bar{J} \ge \frac{1}{\theta} \left\{ \| \delta x(0) \|^2_{P_0^{-1}} + \int_0^{t_s} \delta e^T H^{\,T} \bar{V}^{-1} H \delta e \, dt \right\}$$

$$+ \| \rho_2 \|^2_{\Pi(t_s)}.$$

The right-hand side term can be positive by letting $t_s \rightarrow t_2$ which is a contradiction.

c) Suppose the Assumptions 2a) and 2b) hold but the Assumption 2c) is violated at $t = t_3$, $0 \le t_3 \le T$. Then, for some $\rho_3 \neq 0$, $\rho_3^T \{ P^{-1}(t_3) + \theta \Pi(t_3) \} \rho_3 \le 0$. Choose

$$\delta w = -\theta W \Gamma^T P^{-1} \delta e, \quad \delta v = \Gamma_1^{-1} H \delta e; \qquad \text{for } 0 \le t \le t_3$$

$$\delta w = -\theta W \Gamma^T \Pi \, \delta x, \quad \delta v = 0; \qquad \text{for } t_3 < t \le T$$

and $\delta x(0) \neq 0$ as $\delta x(t_3) = \rho_3$. From Claim 3, $\delta \hat{x}(t) = 0$ and $\delta \bar{u}(t) = 0$ over $[0, t_3]$. By identifying t_s in b) and c) as t_3

$$2 \delta^2 \bar{J} = \frac{1}{\theta} \rho_3^T \{ P^{-1}(t_3) + \theta \Pi(t_3) \} \rho_3$$

$$+ \int_{t_3}^T \| \delta \bar{u} + R^{-1} B^T \Pi \, \delta x \|^2_R \, dt \ge 0$$

which is a contradiction.   □

Remark: Similar results have come to our attention are given in [8], [9].

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Ihnseok Rhee received the B.S. and M.S. degrees in aeronautical engineering from Seoul National University, Seoul, Korea, in 1982 and 1984, respectively, and the Ph.D. degree from the University of Texas, Austin, in 1990.

Currently, he is a senior research engineer for the Korea Aerospace Research Institute, Daeduck Science Town, Daejun, Korea. His current interests are robust control, and optimal control and estimation.

Jason L. Speyer (M'71—SM'82—F'85) received the B.S. degree in aeronautics and astronautics from the Massachusetts Institute of Technology, Cambridge, in 1960, and the Ph.D. degree in applied mathematics from Harvard University, Cambridge, MA, in 1968.

His industrial experience includes research at Boeing, Raytheon, Analytical Mechanics Associated, and the Charles Stark Draper Laboratory. He was the Harry H. Power Professor in Engineering in the Department of Aerospace Engineering and Engineering Mechanics, University of Texas, Austin. Currently, he is a Professor in the Mechanical, Aerospace, and Nuclear Engineering Department at the University of California, Los Angeles. He spent a research leave as a Lady Davis Visiting Professor at the Technion—Israel Institute of Technology, Haifa, Israel, in 1983 and was the 1990 Jerome C. Hunsaker Visiting Professor of Aeronautics and Astronautics at the Massachusetts Institute of Technology.

Dr. Speyer has been an elected member of the Board of Governors of the IEEE Control Systems Society and Chairman of the Technical Committee on Aerospace Controls. He is a Fellow of the American Institute of Aeronautics and Astronautics.

Application of a Game Theoretic Controller to a Benchmark Problem

Ihnseok Rhee* and Jason L. Speyer†
University of Texas at Austin, Austin, Texas 78712

The game theoretic controller whose structure is identical to that of both the linear exponential Gaussian and the H_∞ controller is applied to the problem of controlling a mass-spring system that approximates the dynamics of a flexible structure. By viewing the plant parameter variation as an internal feedback loop, plant uncertainties of the system, input, and output matrices can be decomposed into a fictitious input/output system with unknown gains. These fictitious input/output directions due to parameter uncertainty are used in constructing the gains for the game theoretic controller. The resulting control reduces the effect of parameter uncertainty on the system performance.

I. Introduction

A SYNTHESIS procedure is described for the design of a state feedback control law for a linear time-invariant system in the presence of parameter uncertainty in the system, input, and output matrices. The parameter uncertainty is modeled via an input/output decomposition procedure.[1,2] A differential game approach has been taken for this problem in Ref. 3, where the parameter uncertainty was not decomposed and only the uncertainty in the system matrix is considered. In Refs. 4–6 the Lyapunov stability theory has been used to design a control law for a system with uncertainty. In Refs. 1 and 2, by adopting an input/output decomposition of the parameter uncertainty, the uncertain system is represented as an internal feedback loop (IFL) in which the parameter uncertainty is embedded in the system as a fictitious disturbance. Tahk and Speyer[1,2] developed the parameter robust linear quadratic Gaussian (PRLQG) synthesis procedure, which is an LQG design based on an extension of loop-transfer recovery for the IFL description. In Refs. 1, 2, and 6, the system is augmented to accommodate the input and output matrix uncertainty. In this paper, by considering the input and a fictitious input in the IFL description as two noncooperative players, a finite-time linear differential game problem is constructed based on the results of Ref. 7. By taking the limit to an infinite-time, time-invariant linear system, a time-invariant control law is obtained. It is shown that the resulting time-invariant controller stabilizes the uncertain system for a prescribed parameter uncertainty bound. These results are presented in Sec. II.

This approach is applied in Sec. III to a benchmark problem composed of two masses and a spring with an unknown spring constant. The input is applied to the first mass and a noisy measurement is made of the position of the second mass. Furthermore, a harmonic forcing function of unknown amplitude and phase is applied to the second mass. The objective is to regulate the second mass about the zero position given the assumed uncertainties. A robust compensation is determined that has four nonminimal phase zeros.

II. Game Theoretic Controller

A controller for a linear time-invariant system with parameter uncertainties in the system, input, and output matrices is derived via the differential game framework.

Consider a time-invariant system with uncertainties in system, input, and output matrices described by

$$\dot{x} = (A_0 + \Delta A)x + (B_0 + \Delta B)u \qquad (1)$$

$$z = (H_0 + \Delta H)x \qquad (2)$$

where x, u, and z denote the state vector, the input vector, and the measurement vector, respectively; A_0, B_0, and H_0 denote the nominal system matrix, the nominal input matrix, and the nominal measurement matrix with suitable dimensions, respectively; and ΔA, ΔB, and ΔH are perturbations of the system matrix, the input matrix, and the measurement matrix, respectively, due to parameter variations. It is assumed that (A_0, B_0) is a stabilizable pair and (H_0, A_0) is a detectable pair.

By adopting the input/output decomposition modeling[1,2] of the perturbations, ΔA, ΔB, and ΔH are represented as

$$\Delta A = DL_a(\epsilon)E, \qquad \Delta B = FL_b(\epsilon)G, \qquad \Delta H = YL_h(\epsilon)Z \quad (3)$$

where ϵ denotes the parameter variation vector, which is constant but unknown, and all other matrices are known constant matrices. The elements of ϵ need not be independent of each other.

A. State Feedback

In this subsection all states are assumed to be perfectly measured, and the control u is restricted to a state feedback, i.e., $u = u(x)$.

With the plant perturbation modeling given by Eq. (3), the uncertain dynamic system [Eq. (1)] can be represented as an internal feedback loop[1,2] in which the system is assumed to be forced by fictitious disturbances caused by the parameter uncertainty:

$$\dot{x} = A_0 x + B_0 u + \Gamma_f w_f \qquad (4)$$

$$y_f = \begin{bmatrix} E \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ G \end{bmatrix} u \qquad (5)$$

$$w_f = L(\epsilon)y_f \qquad (6)$$

where $\Gamma_f = [D \; F]$, w_f is the fictitious disturbance, and

$$L(\epsilon) = \begin{bmatrix} L_a(\epsilon) & 0 \\ 0 & L_b(\epsilon) \end{bmatrix}$$

Consider a quadratic performance index,

$$J = \frac{1}{2}\int_0^T |y|^2 \, dt$$

where T is a fixed final time, and

$$y = \begin{bmatrix} C \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ C_1 \end{bmatrix} u$$

Assume that all admissible parameter variations are characterized as

$$\frac{\int_0^T y_f^T L^T(\epsilon) L(\epsilon) y_f^T \, dt}{\int_0^T y_f^T y_f \, dt} = \frac{\int_0^T w_f^T w_f \, dt}{\int_0^T y_f^T y_f \, dt} \le \gamma^2$$

where γ is a positive constant. Note that $|L(\epsilon)|_s \le \gamma$ for all admissible parameter variations, where $|\cdot|_s$ denotes the spectral norm.

For a given control law $u = u(x)$, the performance index J achieves its maximum when the parameter variations are the worst case for the control u. The worst case occurs when w_f uses all of the available control, i.e.,

$$\gamma^{-2}\int_0^T w_f^T w_f \, dt = \int_0^T y_f^T y_f \, dt \tag{7}$$

Consider a control law that minimizes J for the worst case w_f. Then a game situation arises such that

$$\min_u \max_{w_f} J$$

subject to Eqs. (1) and (7). Adjoining the constraint (7) to the performance index J yields the problem of

$$\min_u \max_{w_f} \frac{1}{2}\int_0^T [\rho^2 y^T y + (y_f^T y_f - \gamma^{-2} w_f^T w_f)] \, dt \tag{8}$$

subject to Eq. (4), where ρ is a constant to be determined by trial and error to satisfy Eq. (7). It is well known[3,8] that if there exists a real symmetric solution $\Pi(t)$ over the interval $t \in [0, t_f]$ to the Riccati differential equation (RDE),

$$-\dot{\Pi} = A_0^T \Pi + \Pi A_0 - \Pi(B_0 R^{-1} B_0^T - \gamma^2 \Gamma_f \Gamma_f^T)\Pi + Q_s$$

with the final condition $\Pi(t_f) = 0$, where

$$Q_s = \rho^2 C^T C + E^T E, \qquad R = \rho^2 C_1^T C_1 + G^T G$$

and R is assumed to be positive definite, then the optimal strategies u^* and w_f^* for u and w_f, respectively, are given as

$$u^* = -R^{-1}B_0^T \Pi(t) x$$

$$w_f^* = \gamma^2 \Gamma_f^T \Pi(t) x$$

For the case where $T \to \infty$, if there exists a nonnegative definite solution to the algebraic Riccati equation (ARE),

$$0 = A_0^T \Pi_s + \Pi_s A_0 - \Pi_s(B_0 R^{-1} B_0^T - \gamma^2 \Gamma_f \Gamma_f^T)\Pi_s + Q_s \tag{9}$$
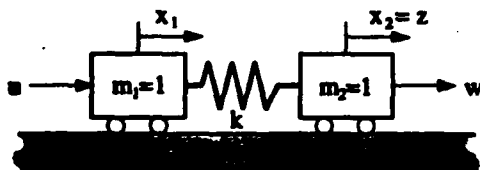

Fig. 1 Mass-spring system.

then $\Pi(t)$ converges to the minimal nonnegative definite solution[3] to the ARE (9).[3,7] Hence, u^* and w_f^* become time-invariant strategies described by

$$u^* = -R^{-1}B_0^T \Pi_s x \tag{10a}$$

$$w_f^* = \gamma^2 \Gamma_f^T \Pi_s x \tag{10b}$$

where Π_s is the minimal nonnegative definite solution to the ARE [Eq. (9)].

In the worst case design, since the fictitious disturbance w_f is not an intelligent player, only the control strategy for the control u given by Eq. (10a) can be implemented.

Claim 1. *Suppose that* $\mathcal{D}^T\mathcal{D} + \mathcal{G}^T\mathcal{G} > 0$ *and let* $\mathcal{U}_1$ *and* $\mathcal{U}_2$ *be arbitrary positive-definite matrices with suitable dimension. Then*

$$\mathcal{D}^T\mathcal{U}_1\mathcal{D} + \mathcal{G}^T\mathcal{U}_2\mathcal{G} > 0$$

Proof. It is sufficient to prove that $\mathcal{D}^T\mathcal{U}_1\mathcal{D} + \mathcal{G}^T\mathcal{U}_2\mathcal{G}$ is nonsingular. Suppose that there exists a nonzero z such that $z^T(\mathcal{D}^T\mathcal{U}_1\mathcal{D} + \mathcal{G}^T\mathcal{U}_2\mathcal{G})z = 0$. Then $\mathcal{D}z = 0$ and $\mathcal{G}z = 0$ since $\mathcal{U}_1$ and $\mathcal{U}_2$ are positive definite; hence, $(\mathcal{D}^T\mathcal{D} + \mathcal{G}^T\mathcal{G})z = 0$, which contradicts the assumption. ∎

Claim 2. *Let* $\mathcal{D}^T\mathcal{D} + \mathcal{G}^T\mathcal{G} = \mathcal{F}^T\mathcal{F}$ *and let* $\mathcal{D}^T\mathcal{D} + \mathcal{G}^T\mathcal{U}\mathcal{G} = \mathcal{F}_1^T\mathcal{F}_1$, *where* $\mathcal{U}$ *is an arbitrary positive-definite matrix with a suitable dimension. If* $(\mathcal{F}, \mathcal{Q})$ *is detectable, then* $(\mathcal{F}_1, \mathcal{Q} + \mathcal{X}\mathcal{G})$ *is detectable for all* $\mathcal{X}$ *with suitable dimensions.*

Proof. Suppose that $(\mathcal{F}_1, \mathcal{Q} + \mathcal{X}\mathcal{G})$ is not detectable. Then there exists a nonzero vector z for some s in the closed right half plane such that $(sI - \mathcal{Q} - \mathcal{X}\mathcal{G})z = 0$ and $\mathcal{F}_1 z = 0$. Since $\mathcal{U} > 0$,

$$z^T(\mathcal{F}_1^T\mathcal{F}_1)z = z^T(\mathcal{D}^T\mathcal{D} + \mathcal{G}^T\mathcal{U}\mathcal{G})z = 0$$

which implies that $\mathcal{D}z = 0$ and $\mathcal{G}z = 0$. Hence,

$$(sI - \mathcal{Q} - \mathcal{X}\mathcal{G})z = (sI - \mathcal{Q})z$$

Therefore,

$$\begin{bmatrix} sI - \mathcal{Q} \\ \mathcal{F} \end{bmatrix} z = 0$$

which contradicts the assumption that $(\mathcal{F}, \mathcal{Q})$ is detectable. ∎

Proposition 1. *Assume that* $R > 0$ *and* $(Q_s^{1/2}, A_0)$ *is a detectable pair. Suppose that for a given* ρ *and* γ *there exists a nonnegative definite solution* Π_s *to the ARE [Eq. (9)]. Then the control law given as*

$$u = -R^{-1}B_0^T \Pi_s x \tag{11}$$

stabilizes the uncertain dynamic system (1) for all ϵ *such that* $|L_a(\epsilon)|_s < \gamma$ *and* $|L_b(\epsilon)|_s < \gamma$.

Proof. By using the control law (11), the closed-loop system is described as

$$\dot{x} = A_s x \tag{12}$$

where

$$A_s = A_0 + DL_a(\epsilon)E - \{B_0 + FL_b(\epsilon)G\}R^{-1}B_0^T\Pi_s$$

The ARE [Eq. (9)] can be rewritten as following the Lyapunov equation:

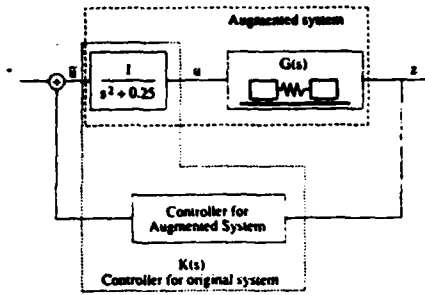$$A_s^T\Pi_s + \Pi_s A_s = -Q_1 \tag{13}$$

Fig. 2 Block diagram of closed-loop system.

where

$$Q_1 = \Pi_s B_0 R^{-1} \Delta_b R^{-1} B_0^T \Pi_s + E^T \Delta_e E + \rho^2 C^T C$$

$$+ \gamma^2 (\Pi_s D - \gamma^{-2} E^T L_a^T)(\Pi_s D - \gamma^{-2} E^T L_a^T)^T$$

$$+ \gamma^{-2} \Pi_s (\gamma^2 F + B_0 R^{-1} G^T L_b^T)(\gamma^2 F + B_0 R^{-1} G^T L_b^T)^T \Pi_s$$

$$\Delta_a = I - \gamma^{-2} L_a(\epsilon)^T L_a(\epsilon)$$

$$\Delta_b = \rho^2 C_1^T C_1 + G^T[I - \gamma^{-2} L_b(\epsilon)^T L_b(\epsilon)]G$$

$\|L_a(\epsilon)\|_s < \gamma$ implies that $\Delta_a > 0$, and $\|L_b(\epsilon)\|_s < \gamma$ and Claim 1 yield $\Delta_b > 0$. Hence, Q_1 is nonnegative definite. Since $(Q_s^{1/2}, A_0)$ is detectable by assumption, it follows from Claim 2 that $[(E^T \Delta_e E + \rho^2 C^T C)^{1/2}, A_0 + DL_a E]$ is detectable. From Lemma 4.1 (Ref. 9), $[(\Pi_s B_0 R^{-1} \Delta_b R^{-1} B_0^T \Pi_s + E^T \Delta_e E + \rho^2 C^T C)^{1/2}, A_s]$ is also a detectable pair. Applying Lemma 4.1 of Ref. 9 again yields that $(Q_1^{1/2}, A_s)$ is detectable. Applying Lemma 4.2 of Ref. 9 to the Lyapunov equation [Eq. (13)] completes the proof. ∎

Note that Proposition 1 holds for any nonnegative solution to the ARE [Eq. (9)]. However, the minimal nonnegative solution $\bar{\Pi}_s$ produces the smallest gain for the control law.

To design the controller (11), the design parameters ρ and γ should be chosen for the ARE [Eq. (9)] to have a nonnegative definite solution. In particular, as the value of ρ increases, system performance improves, whereas as the value of γ increases, stability robustness with respect to parameter variation improves.

B. Measurement Feedback

In this subsection the state is assumed to be partially measured by Eq. (2) and the control u is restricted to be a measurement feedback.

By use of the uncertainty modeling of Eq. (3), system (1) and (2) include an IFL description:

$$\dot{x} = A_0 x + B_0 u + \Gamma_f w_f \tag{14}$$

$$z = H_0 x + Y v_f \tag{15}$$

$$y_f = \begin{bmatrix} E \\ 0 \\ Z \end{bmatrix} x + \begin{bmatrix} 0 \\ G \\ 0 \end{bmatrix} u \tag{16}$$

$$\begin{bmatrix} w_1 \\ w_2 \\ v_f \end{bmatrix} = \begin{bmatrix} L_a(\epsilon) & 0 & 0 \\ 0 & L_b(\epsilon) & 0 \\ 0 & 0 & L_h(\epsilon) \end{bmatrix} y_f \tag{17}$$

where $\Gamma_f = [D \ F]$ and $w_f = [w_1^T \ w_2^T]^T$.

With an approach similar to that taken in Sec. II.A, a differential game, where the fictitious disturbances w_f and v_f and the initial conditions play against the control u, is constructed such that

$$\min_u \max_{w_f} \max_{v_f} \max_{x(0)} \{-\rho_1^2 \{x(0) - \hat{x}_0\}^T \{x(0) - \hat{x}_0\}$$

$$+ \int_0^T \{\rho^2 y^T y + y_f^T y_f - \gamma^{-2}(w_f^T w_f + v_f^T v_f)\} \, dt\}$$

subject to Eqs. (14) and (15), where the cost for the initial conditions is included to handle the uncertainty in the initial condition from the nominal value of $\hat{x}_0$. As $T \to \infty$, a time-invariant controller is obtained in Ref. 7 as

$$\dot{x}_c = A_c x_c + B_c z \tag{18a}$$

$$u = C_c x_c \tag{18b}$$

where

$$A_c = A_0 - B_0 R^{-1} B_0^T \Pi_m - M H_0^T V^{-1} H_0 + \gamma^2 \Gamma_f \Gamma_f^T \Pi_m$$

$$B_c = M H_0^T V^{-1}, \qquad C_c = -R^{-1} B_0^T \Pi_m$$

$$M = (I - \gamma^2 P_m \Pi_m)^{-1} P_m$$

if there exist $\Pi_m \geq 0$ and $P_m > 0$ satisfying the AREs:

$$0 = A_0^T \Pi_m + \Pi_m A_0 - \Pi_m (B_0 R^{-1} B_0^T - \gamma^2 \Gamma_f \Gamma_f^T) \Pi_m + Q_m \tag{19}$$

$$0 = A_0 P_m + P_m A_0^T - P_m (H_0^T V^{-1} H_0 - \gamma^2 Q_m) P_m + \Gamma_f \Gamma_f^T \tag{20}$$

such that $P_m - \gamma^2 \Pi_m > 0$, where

$$Q_m = \rho^2 C^T C + E^T E + Z^T Z, \qquad V = Y Y^T$$

and V is assumed to be positive definite. Since both AREs [(19) and (20)] may have more than one nonnegative definite or positive definite solution, many controllers can be constructed from the formulation (18). Note that $M > 0$ and satisfies the ARE:

$$(A_0 + \gamma^2 \Gamma_f \Gamma_f^T \Pi_m) M + M (A_0 + \gamma^2 \Gamma_f \Gamma_f^T \Pi_m)^T$$

$$- M [H_0^T V^{-1} H_0 - \gamma^2 \Pi_m B_0 R^{-1} B_0^T \Pi_m] M + \Gamma_f \Gamma_f^T = 0 \tag{21}$$

By using the controller (18), the closed-loop system becomes

$$\begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} = (A_d + \Delta A_d) \begin{bmatrix} x \\ x_c \end{bmatrix} \tag{22}$$

$$A_d = \begin{bmatrix} A_0 & B_0 C_c \\ B_c H_0 & A_c \end{bmatrix}, \qquad \Delta A_d = \begin{bmatrix} DL_a E & FL_b GC_c \\ B_c Y L_h Z & 0 \end{bmatrix}$$

and where ΔA_d is the variation of the closed-loop system due to the uncertainty in system (1) and (2).

Proposition 2. Assume that $R > 0$, $V > 0$, $(Q_m^{1/2}, A_0)$ is detectable, and (A_0, Γ_f) is controllable. If there exist $\Pi_m \geq 0$, $P_m > 0$ such that $P_m^{-1} - \gamma^{-2} \Pi_m$, then the controller (18) stabilizes the closed-loop system (22) for all ϵ such that

$$\|L_a(\epsilon)\|_s < \gamma, \qquad \|L_b(\epsilon)\|_s < \gamma, \qquad \|L_h(\epsilon)\|_s < \gamma \tag{23}$$

Proof. Equation (21) can be rewritten as

$$M A_1^T + A_1 M = -Q_1$$

where

$$A_1 = A_0 - MH_0^T V^{-1} H_0 + \gamma^2 \Gamma_f \Gamma_f^T \Pi_m$$

$$Q_1 = M(H_0^T V^{-1} H_0 + \gamma^2 \Pi_m B_0 R^{-1} B_0^T \Pi_m) M + \Gamma_f \Gamma_f^T$$

Since (A_0, Γ_f) is controllable by assumption, it follows from Lemma 4.1 (Ref. 9) that $(A_1, Q_1^{1/2})$ is controllable. Since $M > 0$, it follows from Lemma 4.2 (Ref. 9) that A_1 is stable. It can be verified that $\mathcal{K}$, defined as

$$\mathcal{K} = \begin{bmatrix} P_m^{-1} & -M^{-1} \\ -M^{-1} & M^{-1} \end{bmatrix}$$

satisfies the ARE:

$$0 = A_d^T \mathcal{K} + \mathcal{K} A_d + \mathcal{K} Q_2 \mathcal{K} + \gamma^2 Q_3 \qquad (24)$$

where

$$Q_2 = \begin{bmatrix} \Gamma_f \Gamma_f^T & 0 \\ 0 & B_c V B_c^T \end{bmatrix}, \qquad Q_3 = \begin{bmatrix} Q_m & 0 \\ 0 & C_c^T R C_c \end{bmatrix}$$

Rewriting Eq. (24) as

$$(A_d + \Delta A_d)^T \mathcal{K} + \mathcal{K}(A_d + \Delta A_d) = -Q_4 \qquad (25)$$

where $Q_4 = \mathcal{K} Q_2 \mathcal{K} + \gamma^2 Q_3 - \Delta A_d^T \mathcal{K} - \mathcal{K} \Delta A_d$. Then Q_4 can be written in the following form:

$$Q_4 = \begin{bmatrix} P_m^{-1} F \\ \Pi_m B_0 R^{-1} G^T L_b^T - M^{-1} F \end{bmatrix}$$

$$\times \begin{bmatrix} P_m^{-1} F \\ \Pi_m B_0 R^{-1} G^T L_b^T - M^{-1} F \end{bmatrix}^T$$

$$+ \begin{bmatrix} E^T L_a^T - P_m^{-1} D \\ M^{-1} D \end{bmatrix} \begin{bmatrix} E^T L_a^T - P_m^{-1} D \\ M^{-1} D \end{bmatrix}^T$$

$$+ \begin{bmatrix} Z^T L_h^T + H_0^T V^{-1} Y \\ -H_0 V^{-1} Y \end{bmatrix} \begin{bmatrix} Z^T L_h^T + H_0^T V^{-1} Y \\ -H_0 V^{-1} Y \end{bmatrix}^T$$

$$+ \gamma^2 \begin{bmatrix} \rho^2 C^T C + E^T \Delta_a E + Z^T \Delta_h Z & 0 \\ 0 & \Pi_m B_0 R^{-1} \Delta_b R^{-1} B_0^T \Pi_m \end{bmatrix}$$

where

$$\Delta_h = I - \gamma^{-2} L_h^T L_h$$

Since $\|L_a(\epsilon)\|_s < \gamma$, $\|L_b(\epsilon)\|_s < \gamma$, and $\|L_h(\epsilon)\|_s < \gamma$, Q_4 is nonnegative definite. From Lemma 4.2 (Ref. 9),

$$(\{\rho^2 C^T C + E^T \Delta_a E + Z^T \Delta_h Z\}^{1/2}, A_0 + DL_a E)$$

is detectable since $(Q_m^{1/2}, A_0)$ is detectable and $\Delta_a, \Delta_h > 0$. Therefore, there exists L such that A_2, defined as

$$A_2 \triangleq A_0 + DL_a E - L(\rho^2 C^T C + E^T \Delta_a E + Z^T \Delta_h Z)^{1/2}$$

is stable. Let

$$Q_5 = \begin{bmatrix} (\rho^2 C^T C + E^T \Delta_a E + Z^T \Delta_h Z)^{1/2} & 0 \\ 0 & \Delta_b^{1/2} R^{-1} B_0^T \Pi_m \end{bmatrix}$$
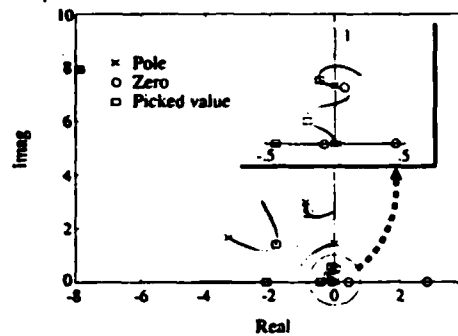
Since $I - \gamma^{-2} L_b^T L_b > 0$ and $R > 0$, it follows from Claim 1 that $\Delta_b > 0$. Observe that, for this L,

$$A_d + \Delta A_d - \begin{bmatrix} L & -(B_0 + FL_b G) \Delta_b^{-1/2} \\ 0 & -B_0 \Delta_b^{-1/2} \end{bmatrix} Q_5$$

$$= \begin{bmatrix} A_2 & 0 \\ MH_0^T V^{-1}(H_0 + YL_b Z) & A_1 \end{bmatrix}$$

Since A_1 and A_2 are stable, $(Q_5, A_d + \Delta A_d)$ is a detectable pair which implies that, by Lemma 4.1 of Ref. 9, $(Q_4, A_d + \Delta A_d)$ is detectable. Since $\mathcal{K}$ is a nonnegative definite matrix, the proof is completed by applying Lemma 4.2 of Ref. 9 to Eq. (25). ∎

Note that the proposition holds for all controllers constructed from the solution of AREs and is therefore very conservative.

To design the controller (18), the design parameters ρ and γ should be choosen for the AREs [(19) and (20)] to have a nonnegative definite solution and a positive definite solution, respectively. In particular, as the value of ρ increases, system performance improves, whereas as the value of γ increases, stability robustness with respect to parameter variation improves.

In the usual case the positivity of V and the controllability of (A, Γ_f) do not hold. However, these can be avoided by redefining V and Γ_f as

$$V = \Gamma_1 \Gamma_1^T + YY^T, \qquad \Gamma_f = [\Gamma \ D \ F]$$

where Γ_1 and Γ are chosen to ensure that $V > 0$ and (A, Γ_f) is controllable. It can be proved with minor change that Proposition 2 holds for these new V and Γ_f.

III.   Two Mass-Spring System

Consider a mass-spring system, shown in Fig. 1, that approximates the dynamics of a flexible structure.[2] The system is described by

$$\ddot{x}_1 + k(x_1 - x_2) = u \qquad (26a)$$

$$\ddot{x}_2 + k(x_2 - x_1) = w \qquad (26b)$$

with a noncollocated measurement

$$z = x_2 \qquad (27)$$

where k is an unknown constant with nominal value $k_0 = 1$, u is an actuator input, and w is a cyclic disturbance described by

$$w(t) = A_w \sin(0.5t + \varphi)$$

where A_w and φ are constant but unknown. The transfer function form of the system and measurement equations, Eqs. (26) and (27), respectively, is given as

$$G(s) \triangleq \frac{z(s)}{u(s)} = \frac{1}{s^2(s^2 + 2)}$$

The design objective is to regulate x_2 and to reject the external cyclic disturbance in x_2 for all k with $0.5 < k < 2$.

To handle the cyclic disturbance, differentiate Eq. (26) until w disappears in the resulting system. Differentiating Eq. (26) twice yields

$$x_1^{(IV)} = -k(\ddot{x}_1 + 0.25x_1 - \ddot{x}_2 - 0.25x_2) - 0.25\ddot{x}_1 + \bar{u} \quad (28a)$$

$$x_2^{(IV)} = -k(\ddot{x}_2 + 0.25x_2 - \ddot{x}_1 - 0.25x_1) - 0.25\ddot{x}_2 \quad (28b)$$

where the parenthetical superscripts represent the time-derivative order and $\bar{u}$ is a new control variable defined as

$$\bar{u} = \ddot{u} + 0.25u \quad (29)$$

where

$$A_c = \begin{bmatrix} 0 & 1 & 8.41 & 0 & 0 & 0 \\ -147.26 & -17.11 & 60.32 & -61.00 & 44.21 & -174.52 \\ 0 & 0 & -7.19 & 1 & 0 & 0 \\ 0 & 0 & -25.78 & 0 & 1 & 0 \\ 0 & 0 & -51.65 & 0 & 0 & 1 \\ 1.03 & 0 & -41.20 & 0.11 & -1.11 & 0.30 \end{bmatrix}$$

$$B_c = [-8.41 \quad -37.85 \quad 7.19 \quad -25.78 \quad 51.65 \quad 40.40]^T$$

$$C_c = [-146.23 \quad -17.11 \quad 22.24 \quad -60.89 \quad 43.34 \quad -174.22]$$

The new system (28) contains uncontrollable poles at $s = \pm 0.5j$. To remove the uncontrollable poles from Eqs. (28), a new state, ξ, is introduced as $\xi = \ddot{x}_1 + 0.25x_1$. Then Eqs. (28) are represented in terms of x_2 and ξ as

$$\ddot{\xi} = -k\xi + k(\ddot{x}_2 + 0.25x_2) + \bar{u} \quad (30a)$$

$$x_2^{(IV)} = -(k + 0.25)\ddot{x}_2 - 0.25kx_2 + k\xi \quad (30b)$$

A controller is designed for this augmented system. Figure 2 shows that the controller for original system is constructed by combining the controller for the augmented system (30) and the relation (29). Define

$$x = [\xi \quad \dot{\xi} \quad x_2 \quad \dot{x}_2 \quad \ddot{x}_2 \quad x_2^{(III)}]^T$$

Then Eqs. (27) and (30) can be represented in state-space form as

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -k & 0 & 0.25k & 0 & k & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ k & 0 & -0.25k & 0 & -k-0.25 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \bar{u}$$

$$\underbrace{}_{A} \quad \underbrace{}_{B}$$

$$z = \underbrace{[0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0]}_{H} x$$

The variation of system matrix due to the uncertainty of k can be decomposed as

$$\Delta A = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}}_{D} \left[\frac{\Delta k}{n} \right] \underbrace{\left[-n \quad 0 \quad 0.25n \quad 0 \quad n \quad 0 \right]}_{E}$$

With choices of

$$C = H, \quad \Gamma = 0.07 \cdot [0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1]^T, \quad \Gamma_1 = 0.033$$

$$C_1 = 0.08, \quad \rho = 1, \quad \gamma = 0.043, \quad n = 10$$

the control $\bar{u}$ can be obtained by using Eq. (18) as

$$\dot{x}_c = A_c x_c + B_c z, \quad \bar{u} = C_c x_c \quad (31)$$

Note that n is a weighting between the E direction and C, and Γ is chosen to ensure that (A, Γ_f) is controllable (see Proposition 2). The design parameters ρ, γ, and n were chosen to satisfy the robustness requirement that $0.5 < k < 2$ and the transient requirement that the system settle within 20 s. The minimal nonnegative definite solutions for the AREs [(19) and (20)] are used in controller design. Combining Eqs. (29) and (31) yields an eighth-order controller for the original system (28) in the form of

$$\dot{x}_c = \begin{bmatrix} A_c & 0_{6 \times 2} \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} C_c & \begin{bmatrix} 0 & 1 \\ -0.25 & 0 \end{bmatrix} \end{bmatrix} \dot{x}_c + \begin{bmatrix} B_c \\ 0_{2 \times 1} \end{bmatrix} z$$

$$u = [0_{1 \times 6} \quad 1 \quad 0] \dot{x}_c$$

or in the transfer function form:

$$K(s) \triangleq \frac{u(s)}{z(s)} = \frac{-4430(s + 0.08)(s - 0.44)(s - 2.83)}{(s^2 + 0.25)(s^2 + 1.78s + 9.67)}$$

$$\times \frac{(s^2 - 0.1s + 0.24)}{(s^2 + 6.56s + 13.51)(s^2 + 15.68s + 124.99)}$$

where the compensator poles are at $\pm 0.5j$, $-7.84 \pm 7.97j$, $-3.28 \pm 1.66j$, and $-0.89 \pm 2.98j$, and the complex compensator zeros are at $0.05 \pm 0.49j$. The zero configuration of the compensator represents nonminimum compensation. The closed-loop poles are at -2.14, -0.33, $-0.13 \pm 0.56j$, $-0.19 \pm 0.25j$, $-0.77 \pm 2.50j$, $-1.83 \pm 1.45j$, and $-7.84 \pm 7.97j$.
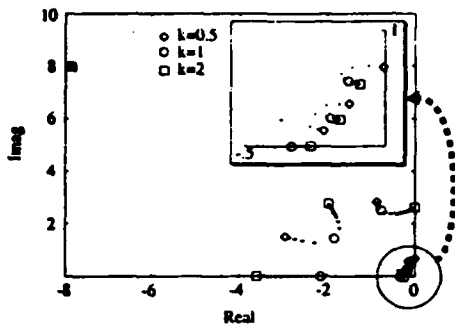
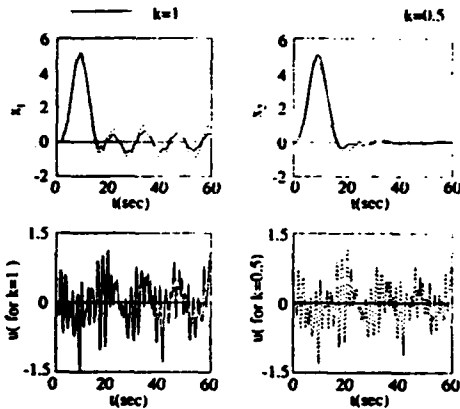**Fig. 4   Root locus of closed-loop system with respect k.**

Fig. 5   Time response.

Figure 3 shows the root locus of $1 - \alpha G(s)K(s)$ with respect to α. For the controller $K(s)$, the gain margin is 3.23 dB at the frequency 0.67 rad/s, and the phase margin is 25 deg at the frequency 0.20 rad/s. The values of γ and n show that the stability margin $|\Delta k| < 0.43$ is guaranteed for this compensator. However, the root locus, shown in Fig. 4, shows that the compensator stabilizes the system over $0.5 < k < 2$. Note that the compensator pole at $-7.84 \pm 7.97j$ is unaffected by parameter or open-loop gain changes. The removal of these poles from the compensator does not affect stability robustness or transient response.

In the simulation the measurement is assumed corrupted by zero mean white Gaussian noise with a power spectral density

of (0.33).[2] Time responses, shown in Fig. 5, for the nominal system and the perturbed system with $k = 0.5$ are simulated. For both simulations, $A_w = 0.5$, $\varphi = 0$, and all initial conditions are zero. Figure 5 shows that, for the nominal case, the controlled variable x_2 has settled down and the cyclic disturbance is rejected in x_2 in ~ 20 s, and for the perturbed system with $k = 0.5$, the settling time has been delayed.

IV.   Conclusions

A game theoretic controller was applied to a mass-spring system disturbed by cyclic external force. The cyclic disturbance was augmented to the system by a procedure involving differentiation and transformation. The resulting state and control are used to design the game theoretic compensator. A nonminimum phase compensator resulted.

Acknowledgment

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an elimination method such as the cyclic reduction method. This line of reasoning should provide a further clue to improve present parallel and sequential algorithms for engineering computation by exploiting specific system structures.

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System Characterization of Positive Real Conditions

H. Weiss, Q. Wang, and J. L. Speyer

Abstract—Necessary and sufficient conditions for positive realness in terms of state-space matrices are presented under the assumption of complete controllability and complete observability of square systems with independent inputs. By a particular transform of these conditions, a direct algorithm for testing positive realness is determined that requires only checking a set of simple algebraic conditions. This provides an alternative procedure to the positive real lemma and to the s-domain inequalities. Based on this algorithm, a synthesis of a positive real system via output feedback is presented.

I. INTRODUCTION

Positive real systems play a major role in control theory, especially in adaptive control, and in stability analysis. The impressive development of adaptive control and self-tuning regulation over the last two decades [1], [2] is hinged on satisfaction of some positive realness conditions. Alternatively, considerable initial knowledge about the controlled plant must be given. The prior knowledge is used to implement reference models, iden_crs, or observer-based controllers of about the same order as the plant. Since the prior assumptions about the controlled plant may never be entirely satisfied, the stability properties of the related adaptive schemes are debatable. Therefore, a direct adaptive control procedure which does not use identifier or observer-based controllers in the feedback loop is preferred. The implementation of such an algorithm requires positive real controlled plants or alternatively, a synthesis of a positive real plant on the basis of the actual plant.

The existing tools for analysis and synthesis of positive real systems are based in the s-domain on complex variable inequalities which are inconvenient or in the state space requiring the positive real lemma equations. These tools are computationally complex and there is a need for an easily used complementary tool. Necessary and sufficient conditions for positive real systems equivalent to the positive real lemma are developed in Sections II and III using optimal control theory for the associated partially singular problem. Positive real systems are characterized in terms of the necessary and sufficient conditions of optimal control theory such as the generalized Legendre-Clebsch condition [3], [4]. By using a special transformation, the resulting test for positive realness reduces to testing certain square matrices for positive definiteness related to the generalized Legendre-Clebsch condition and the solution to an algebraic Riccati equation of possibly reduced dimension. This test for positive realness parallels that given in [5]. The transformation used here based on the results of [6] produce interesting characterizations of positive real system in terms of system zeros and the necessary conditions in singular optimal control. That a positive real system is of minimum phase becomes transparent in this development. In Section IV, it is proved that if a square system is minimum phase with certain positive real characteristics, there exists a constant output feedback gain such

that the resulting closed-loop system is positive real. Some examples are shown in Section V to illustrate the theory. Concluding remarks are given in Section VI.

The derivation of this new test for positive realness is based upon a state-space formulation of dissipative systems. Basic definitions and physical characteristics are presented below.

A. Dissipative System

Consider the system input–output description $H: U \to Y$ where $U = L_2^l(R_+)$ and $Y = L_2^m(R_+)$. The notation $L_2^l(R_+)$ is used to denote the space of square integrable functions $f: R_+ \to R^l$ where $R_+ = [t_0, \infty)$. The supply rate associated with this system is defined as the function $w: R^l \times R^m \to R$ where

$$w(u, y) = y'Qy + 2y'Su + u'Ru \qquad (1.1)$$

and $Q \in R^{m \times m}$, $S \in R^{m \times l}$, $R \in R^{l \times l}$ are constant matrices, with Q and R symmetric.

Definition 1.1 [7]: A dynamical system H is *dissipative* with respect to the supply rate $w(x, y)$ if and only if

$$\int_{t_0}^{t_1} w[u(t), y(t)] \, dt \geq 0 \qquad (1.2)$$

for all $t_1 \geq t_0$ and all $u \in L_2^l$, whenever the initial state satisfies $x(t_0) = 0$. The concept of a supply rate is related in the general case to the "stored energy" for the system.

Remark 1.1: Passivity corresponds to dissipativeness where $Q = R = 0$, $1 = m$, $S = (1/2)I_m$ and I_m is $m \times m$ identity matrix.

Remark 1.2: Positive realness corresponds to passivity where the dynamical system is linear and time invariant.

Assume that the system under consideration is linear and time-invariant, giving

$$\dot{x} = Ax + Bu \qquad (1.3)$$

$$y = Cx + Du \qquad (1.4)$$

where $x \in R^n$, $u \in R^l$, $y \in R^m$ and A, B, C, and D are constant matrices with appropriate dimensions.

B. Review of the Positive Real Property

The positive real property is related directly to the transfer function matrix description of the system. The positive real lemma, presented in Section II, connects the positive realness to the parameters of a system realization with complete controllability and complete observability.

The Positive Real Property [8, p. 51]: Let $G(s)$ be an $m \times m$ matrix of functions of a complex variable s. Then $G(s)$ is termed positive real if the following conditions are satisfied:

i) All the elements of $G(s)$ are analytic in $\text{Re}[s] > 0$.

ii) $G(s)$ is real for real positive s.

iii) $G^*(s) + G(s) \geq 0$ for $\text{Re}[s] > 0$ where $(\cdot)^*$ denotes complex conjugate transpose.

Remark 1.3: If $G(s)$ is a real rational matrix of functions of s, then necessary and sufficient conditions for the positive real property to hold are given by the following theorem.

Theorem 1.1 [8, p. 53]: Let $G(s)$ be a real rational matrix of functions of s. Then, $G(s)$ is positive real if and only if

i) No element of $G(s)$ has a pole in $\text{Re}[s] > 0$.

ii) $G^*(j\omega) + G(j\omega) \geq 0$ for all real ω, with $j\omega$ not a pole of any element of $G(s)$.

iii) If $j\omega_0$ is a pole of any element of $G(s)$, it is at most a simple pole, and the residue matrix,

$$k = \begin{cases} \lim_{s \to j\omega}(s - j\omega_0)G(s) & \text{if } j\omega_0 \text{ is finite} \\ \lim_{s \to j\omega} G(j\omega)/j\omega & \text{if } j\omega_0 \text{ is infinite} \end{cases}$$

is nonnegative definite Hermitian.

Remark 1.4: Since this test is done for all frequencies, the amount of computation for matrix systems may be large. The approach here only requires a finite number of operations which guarantee positive realness.

II. RELATIONS BETWEEN OPTIMAL CONTROL AND POSITIVE REALNESS

A. The Related Variational Problem

Consider minimizing the cost functional

$$V[x_0, t_0, u(\cdot)] = \int_{t_0}^{t_1} w[u(t), y(t)] \, dt \qquad (2.1)$$

where the supply rate is

$$w(u, y) = y'u = \frac{1}{2}(u'Ru + 2x'C'u) \qquad (2.2)$$

subject to the dynamic system (1.3) and (1.4), where $R = D + D'$. The dimension of u and y is assumed to be m. Denote $u^*(\cdot) \in U$ as the control which minimizes (2.1) subject to the dynamic equation (1.3) and (1.4), where $x(t_0) = x_0$ is prescribed. The necessary and sufficient conditions for $V^*[x_0, t_0] = V[x_0, t_0, u^*(\cdot)]$ to be bounded from below are equivalent to the necessary and sufficient conditions for $V[0, t_0, u(\cdot)]$ to be nonnegative definite (positive real).

Remark 2.1: If $R \geq 0$, and $\text{rank}(R) = r < m$, there exists an orthogonal transformation matrix $\Gamma = [\Gamma_1, \Gamma_2]$ such that

$$\begin{bmatrix} \Gamma_1' \\ \Gamma_2' \end{bmatrix} R[\Gamma_1, \Gamma_2] = \begin{bmatrix} R_r & 0 \\ 0 & 0 \end{bmatrix} \qquad (2.3)$$

where R_r is positive [9]. Positive realness of $\bar{G}(s) = \Gamma'G(s)\Gamma$ is not affected by this transformation.

B. Positive Real Lemma Equations

Necessary and sufficient conditions for $V^*[x_0, t_0]$ to be bounded from below over a finite time interval $[t_0, t_1]$ are presented in [10, theorem II.3.3]. The required positive real conditions are obtained via the extension of Theorem II.3.3 to the time-invariant, infinite-time interval case [11].

Under the complete controllability and complete observability assumption of system (1.3) and (1.4), necessary and sufficient conditions for the nonnegativity of $V[0, t_0, u(\cdot)]$ are that there exist $\pi < 0$, L, and W such that

$$\begin{bmatrix} \pi A + A'\pi & \pi B + C' \\ B'\pi + C & R \end{bmatrix} = \begin{bmatrix} L' \\ W' \end{bmatrix}[L, W] \geq 0 \qquad (2.4)$$

where W and L are of proper dimensions.

By identifying $P = -\pi$, the positive real lemma is stated below.

The Positive Real Lemma [8, p. 218]: Let $G(s)$ be an $m \times m$ matrix of real rational functions of a complex variable s, with $G(\infty) < \infty$. Let $\{A, B, C, D\}$ be a minimal realization of $G(s)$. Then, $G(s)$ is positive real if and only if there exist real matrices P, L, and W with P symmetric and positive definite, such that

$$PA + A'P = -L'L \qquad (2.5)$$

$$B'P = C - W'L \qquad (2.6)$$

$$W'W = D + D'. \qquad (2.7)$$

Remark 2.2: The generalized Legendre–Clebsch condition which is a necessary condition for $V^*[x_0, t_0] > -\infty$ in the totally singular case, for linear time-invariant system as given in [3], is

$$\frac{\partial}{\partial u}(\dot{H}u) = CB - (CB)' = 0 \qquad (2.8)$$

$$\frac{\partial}{\partial u}(\ddot{H}u) = CAB + (CAB)' \leq 0 \qquad (2.9)$$

where H is the variational Hamiltonian and $\lambda \in R^n$ is the associated Lagrange multiplier defined by

$$H = u'Cx + \lambda'(Ax + Bu), \qquad \dot{\lambda} = -H_x.$$

By letting $D = 0$, (2.8) and (2.9) are also obtained from the positive real lemma. (2.8) can be obtained from (2.6) which is in this case $B'P = C$. (2.9) can be obtained by pre- and post- multiplying (2.5) by B' and B, respectively, then applying $B'P = C$.

III. POSITIVE REAL CONDITIONS IN TERMS OF STATE-SPACE MATRICES

Let $\{A, B, C, D\}$ be a minimal realization of $\tilde{G}(s) = \Gamma'G(s)\Gamma$ [see (2.3)]. In terms of state-space matrices, (2.4) gives necessary and sufficient conditions for a positive real system. In this section, necessary and sufficient conditions for positive real systems are developed using singular optimal control [10].

To simplify the notation, we assume that B, C, and D admit the following partition

$$B = [B_r, B_s], \qquad C = \begin{bmatrix} B_r \\ C_s \end{bmatrix}, \qquad R = D + D' = \begin{bmatrix} R_r & 0 \\ 0 & 0 \end{bmatrix}$$

where B_r is an $n{\times}r$ matrix, B_s is an $n{\times}s$ matrix, C_r is an $r{\times}n$ matrix, C_s is an $s{\times}n$ matrix, R is an $m{\times}m$ matrix, and R_r is an $r{\times}r$ matrix, where $r = \text{rank}(R)$ is the dimension of the nonsingular control, and $s = m - r$ is the dimension of the singular control.

R being positive semidefinite is a necessary condition for (2.4) to be satisfied. If $R > 0$, (2.4) can be reduced to a condition based upon a Riccati equation. That is, there exists a symmetric negative definite solution π to the algebraic Riccati equation

$$\pi(A - BR^{-1}C) + (A - BR^{-1}C)'\pi$$
$$-\pi BR^{-1}B'\pi - C'R^{-1}C = 0. \quad (3.1)$$

If R is singular, (2.4) can be rewritten with some matrix V as

$$\begin{bmatrix} \pi A + A'\pi & \pi B_r + C_r & B_s'\pi + C_s \\ B_r'\pi + C_r & R_r & 0 \\ B_s'\pi + C_s & 0 & 0 \end{bmatrix} = V'V$$

or, equivalently, there exist a $\pi < 0$ and some matrix V_r such that

$$\pi B_s + C_s' = 0 \qquad (3.2)$$

and

$$\begin{bmatrix} \pi A + A'\pi & \pi B_r + C_r \\ B_r'\pi + C_r & R_r \end{bmatrix} = V_r'V_r. \qquad (3.3)$$

Under conditions (3.2) and (3.3), two cases are treated separately.

Case 1) If the dimension of the state is less than or equal to the dimension of the singular control, i.e., $n \leq s$, π can be determined from (3.2). If and only if a solution $\pi < 0$ can be solved from (3.2) and the same π satisfies (3.3), the system is positive real.

Case 2) If $n > s$, (3.2) and π being symmetric and negative definite imply that

$$C_sB_s = (C_sB_s)' = -B_s'\pi B_s > 0. \qquad (3.4)$$

Note that (3.4) is necessary, but not sufficient. See (2.8) for a variational interpretation. The new necessary and sufficient conditions are derived for Case 2) in the next subsection.

A. Derivation of Transformed Necessary and Sufficient Conditions

For (partially) singular problem, (3.2) and (3.3) serve as necessary and sufficient conditions for positive realness of a system. Since (3.2) and (3.4) partially determine the structure of π, the original problem can be transformed into the positive realness of a reduced-order system.

By assuming that (3.4) holds, one can perform the following transformation with

$$T = \begin{bmatrix} N \\ C_s \end{bmatrix}, \qquad T^{-1} = [M, B_s(C_sB_s)^{-1}]$$

where N and M are $(n-s){\times}s$ and $s{\times}(n-s)$ matrices consist of basis of the null spaces of B_s and C_s, respectively, such that

$$NB_s = 0, \qquad C_sM = 0, \qquad NM = I_{n-s}.$$

The new realization of $\tilde{G}(s)$ becomes $\{A_T, B_T, C_T, D\}$, where

$$A_T = TAT^{-1} = \begin{bmatrix} NAM & NAB_s(C_sB_s)^{-1} \\ C_sAM & C_sAB_s(C_sB_s)^{-1} \end{bmatrix}$$

$$B_T = TB = [TB_r, TB_s] = \begin{bmatrix} B_{T1} & 0 \\ B_{T2} & C_sB_s \end{bmatrix}$$

$$C_T = CT^{-1} = \begin{bmatrix} C_rT^{-1} \\ C_sT^{-1} \end{bmatrix} = \begin{bmatrix} C_{T1} & C_{T2} \\ 0 & I_{m-r} \end{bmatrix}.$$

By applying (3.2) and (3.3) with respect to $\{A_T, B_T, C_T, D\}$, we obtain

$$\pi \begin{bmatrix} 0 \\ C_sB_s \end{bmatrix} + \begin{bmatrix} 0 \\ I_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (3.5)$$

and

$$\begin{bmatrix} \pi A_T + A_T'\pi & \pi TB_r + (C_rT^{-1})' \\ (TB_r)'\pi + C_rT^{-1} & R_r \end{bmatrix} \geq 0. \qquad (3.6)$$

To satisfying (3.5), the structure of π can only be

$$\pi = \begin{bmatrix} \pi_1 & 0 \\ 0 & -(C_sB_s)^{-1} \end{bmatrix}. \qquad (3.7)$$

In order that $\pi < 0$, the $n - s$ dimensional π_1 must be negative definite.

By substituting (3.7) into (3.6), we obtain

$$\begin{bmatrix} \pi_1A_1 + A_1'\pi_1 & \pi_1B_1 + C_1' \\ TB_1'\pi_1 + C_1 & R_1 \end{bmatrix} \geq 0 \qquad (3.8)$$

where

$$A_1 = NAM \qquad (3.9)$$

$$B_1 = [NAB_s(C_sB_s)^{-1}, NB_r] \qquad (3.10)$$

$$C_1 = \begin{bmatrix} -(C_sB_s)^{-1}C_sAM \\ C_rM \end{bmatrix} \qquad (3.11)$$

and (3.12) as shown at the bottom of the page. According to (2.4) or the positive real Lemma, the proof of positive realness of the original system is reduced to the proof of positive realness of the reduced-order system $\{A_1, B_1, C_1, R_1/2\}$. The process will be continued until either R_1 becomes nonsingular, or, the dimension of A_1 is less than or equal to the dimension of null space of R_1. Furthermore, from (3.8), the upper left-hand block of (3.12) must be nonnegative definite. Note that this satisfies the generalized Legendre–Clebsch condition given in (2.9).

$$R_1 = \begin{bmatrix} -(C_sB_s)^{-1}[C_sAB_s + (C_sAB_s)'](C_sB_s)^{-1} & (C_sB_s)^{-1}(B_s'C_r, -C_sB_r) \\ (C_rB_s - B_r'C_s')(C_sB_s)^{-1} & R_r \end{bmatrix}. \qquad (3.12)$$

Remark 3.1: The calculation of N and M is well defined. Compute the singular value decomposition of B_s and C_s as

$$B_s = [U_{11}, U_{12}]\begin{bmatrix} \Sigma_B \\ 0 \end{bmatrix} V_1'. \qquad C_s = U_2[\Sigma_C, 0]\begin{bmatrix} V_{21}' \\ V_{22}' \end{bmatrix}$$

where V_1, U_2, $[U_{11}, U_{12}]$, and $\begin{bmatrix} V_{21}' \\ V_{22}' \end{bmatrix}$ are all orthogonal matrices, and Σ_B and Σ_C are nonsingular. Then, we can define N and M as

$$N = U_{12}', \qquad M = V_{22}(U_{12}'V_{22})^{-1}.$$

A. Necessary and Sufficient Conditions for Positive Realness

The new necessary and sufficient conditions for positive realness is summarized in the following theorem.

Theorem 3.1: The transfer function $G(s) = \{A, B, C, D\}$ is positive real if and only if

i) $R \geq 0$

ii) If $R > 0$, there exists a positive definite solution P to the following algebraic Riccati equation

$$P(A - BR^{-1}C) + (A - BR^{-1}C)'P + PBR^{-1}B'P$$
$$+ C'R^{-1}C = 0.$$

iii) If rank $R = r < m$, and $n \leq s$, there exists $P = CB(BB) > 0$ satisfying $PB = C'$, and

$$\begin{bmatrix} -PA - A'P & -PB_r + C_r' \\ -B_r'P + C_r & R_r \end{bmatrix} \geq 0.$$

iv) If rank $R = r < m$ and $s < n$, then $C_sB_s = (C_sB_s)' > 0$ and $\{A_1, B_1, C_1, R_1/2\}$ is positive real where A_1, B_1, C_1, and R_1 are defined in (3.9)–(3.12).

Condition ii) is obtained by identifying P with $-\pi$ in (3.1). Condition iii) is the interpretation of (3.2) and (3.3) for the case $n \leq s$. If $P = -\pi > 0$ exists, then $PB_s = C_s'$ and $PB_sB_s' = C_s'B_s$, and $P = C_sB_s(B_sB_s')^{-1} > 0$. Condition iv) corresponds to the situation discussed in Section III-A.

Remark 3.2: If $G(s)$ is strictly proper, and CB is nonsingular, then the eigenvalues of $A_1 = NAM$ are the transmission zeros of system (1.3) and (1.4). Since the system's zero s are determined from

$$\det\left\{\begin{bmatrix} A - sI & B \\ C & 0 \end{bmatrix}\right\} = 0 \qquad (3.13)$$

pre- and post-multiply $\begin{bmatrix} A - SI & B \\ C & 0 \end{bmatrix}$ by nonsingular matrices $\begin{bmatrix} U_{12}' & 0 \\ U_{11}' & 0 \\ 0 & I \end{bmatrix}$ and $\begin{bmatrix} V_{22}'(U_{12}'V_{22})^{-1} & V_{21}'(U_{12}'V_{22})^{-1} & 0 \\ 0 & 0 & I \end{bmatrix}$ respectively, the following matrix is obtained

$$\begin{bmatrix} N(A - sI)M & \cdots & 0 \\ \cdots & \cdots & \Sigma_B V_1' \\ 0 & U_2\Sigma_C(U_{12}'V_{22})^{-1} & 0 \end{bmatrix}.$$

Since $\Sigma_B V_1'$ and $U_2\Sigma_C(U_{12}'V_{22})^{-1}$ are nonsingular, (3.13) is equivalent to

$$\det(N(A - sI)M) = \det(NAM - sI) = 0.$$

That is, the eigenvalues of NAM are the same as the system's finite zeros.

Remark 3.3: From (3.9) and Remark 3.2, we conclude that there are $n - m$ finite zeros for a positive real strictly proper system and all the zeros lie in the closed left-half complex plane. In other words, the system is minimum phase.

To simplify the approach further, a transformation can always be found such that if $n > m$, a minimal realization of $G(s), \{A, B, C, D\}$ can be established so that $C = [0, I]$. In particular, this is accomplished by finding a realization in observability canonical form [11]. In this case, M and N can simply be chosen as

$$M = \begin{bmatrix} I_{n-s} \\ 0 \end{bmatrix}, \qquad N = [I_{n-s}, \; -B_{s1}B_{s2}^{-1}]$$

where $B_s' = [B_{s1}', B_{s2}']$ and B_{s2} is assured invertible.

IV. SYNTHESIS OF POSITIVE REAL SYSTEM VIA OUTPUT FEEDBACK

Consider the linear system

$$\dot{x} = Ax + Bu \qquad (4.1)$$
$$y = Cx. \qquad (4.2)$$

It can be shown that if $CB = (CB)' > 0$, and the system is strictly minimum phase, then an output feedback gain K can be found such that with $u = u_1 + Ky$, the closed-loop system

$$\dot{x} = (A - BK)x + Bu_1 \qquad (4.3)$$
$$y = Cx \qquad (4.4)$$

is positive real with respect to u_1 now as the input. By applying the new positive real conditions to a system $\{A - BKC, B, C, 0\}$, the selection of K becomes straight forward.

Let N and M be such that $NB = 0$ and $CM = 0$. Then the reduced-order system $\{A_1, B_1, C_1, R_1/2\}$ is the one with

$$A_1 = N(A - BKC)M = NAM$$
$$B_1 = N(A - BKC)B(CB)^{-1} = NAB(CB)^{-1}$$
$$C_1 = -(CB)^{-1}C[C(A - BKC)M = -(CB)^{-1}CAM$$
$$R_1 = -(CB)^{-1}[C(A - BKC)B + B'(A - BKC)'C'](CB)^{-1}$$
$$= -(CB)^{-1}[CAB + B'A'C'](CB)^{-1} + K + K'.$$

Provided that $CB = (CB)' > 0$, the positive realness of $\{A_1, B_1, C_1, R_1/2\}$ implies the positive realness of the closed-loop system. By applying the positive real Lemma, it is shown that for some K, there exists a $P_1 > 0$ such that

$$P_1A_1 + A_1'P_1 = -L'L \qquad (4.5)$$
$$P_1B_1 - C_1' = -L'W \qquad (4.6)$$
$$- (C_1B_1)^{-1}[C_1A_1B_1 + B_1'A_1'C_1'](C_1B_1)^{-1}$$
$$+ K + K' = W'W. \qquad (4.7)$$

Since the original system is strictly minimum phase, by choosing L such that (A_1, L) is completely observable, (4.1) admits a solution $P_1 > 0$. In order that W is solvable from (4.6), L can be chosen as a nonsingular square matrix which also guarantees that (A_1, L) be observable. Finally, K can be solved from (4.7). The above discussion is summarized into the following theorem.

Theorem 4.1: If the system is minimum phase, and $(CB) = (CB)' > 0$, there exist a constant feedback matrix K such that the closed-loop system with $u = u_1 - Ky$ is positive real.

V. EXAMPLES

Theorem 3.1 introduces a procedure for testing positive real systems, requires only the testing a series of matrices $C_{i*}B_{i*} > 0$, for $i = 0, 1, 2, \cdots, k$, and the solution to an algebraic Riccati equation $P_i > 0$, where i is the index associated with the new system obtained from the ith iteration, and $i = 0$ corresponds to B_*, C_*, and P. The testing stops when R_i becomes nonsingular, or $\dim(A_i) = 0$.

The following examples illustrate the application of Theorem 3.1.

Example 5.1: $G(s) = \begin{bmatrix} \frac{s^2+s-2}{s^2+2s+2} & \frac{s+2}{s^2+2s+2} \\ \frac{-5}{s^2+2s+2} & \frac{s+3}{s^2+2s+2} \end{bmatrix}$.

A minimal realization of $G(s)$ is $\{A, B, C, D\}$ where

$$A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, $C_* = [2, 1]$, $B_* = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. In the test, $R \geq 0$, $C_*B_* = 1 > 0$.

By choosing $M = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $N = [1, 0]$, we obtain $A_1 = -3$, $B_1 = [1, 1]$, $C_1 = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$, and $R_1 = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$. Since R_1 is nondefinite because the Legendre–Clebsch condition fails, the system is not positive real.

Example 5.2: $G(s) = ((s + 2)(s + 3)/s(s + 1)(s + 4)) = (s^2 + 5s + 6/s^3 + 5s^2 + 4s)$. An observable realization of $G(s)$ is given below

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -4 \\ 0 & 1 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}, \quad C = [0, 0, 1], \quad D = 0.$$

In the test, $R = 0$, $C_*B_* = CB = (CB)' = 1 > 0$. By choosing

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -5 \end{bmatrix},$$

we obtain

$$A_1 = \begin{bmatrix} 0 & -6 \\ 1 & -5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad C_1 = [0, -1], \quad R_1 = 0.$$

Now test positive realness of $\{A_1, B_1, C_1, 0\}$. Since $C_1B_1 = -2 < 0$, the system is not positive real.

Example 5.3: In this example, we are going to construct an output feedback gain K for the system given in Example 5.2 such that the closed-loop system is positive real. From (4.5), by choosing L to be the identity matrix, we obtain $P_1 = \begin{bmatrix} 0.55 & -0.5 \\ -0.5 & 0.8 \end{bmatrix}$. By substituting P and L into (4.6), W is solved as $W = \begin{bmatrix} 1 \\ -2.6 \end{bmatrix}$. By substituting W into (4.7), $K = 6.38$ is obtained.

VI. SUMMARY AND CONCLUSIONS

This paper reviews positive real systems as a subclass of dissipative systems and states the positive real lemma equations. By using the variational problem associated with the partially singular problem, necessary and sufficient conditions for a system to be positive real are stated which are equivalent to the positive real lemma. The Legendre–Clebsch conditions and the zero structure are particularly transparent through the transformation discussed in

Section III-B. These positive realness conditions are expressed in terms of the state-space matrix inequalities and an algebraic Riccati equation of possibly reduced dimension. These conditions do not deal with inequalities tested over the frequency domain or with searching for matrices that satisfy the positive real lemma equations. Essentially, the direct test developed here provide a methodology for using the positive real lemma. A system either satisfies these conditions or does not. These conditions also made the synthesis of positive real system straight forward. Examples are given which demonstrate the power of this approach.

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Stochastic Monotonicity and Concavity Properties of Rate-Based Flow Control Mechanisms

Kenneth C. Budka

Abstract— Using sample path comparisons, we study the stochastic monotonicity and concavity properties of the moving and jumping window flow control mechanisms, leaky bucket flow control mechanism, and the token bank rate control throttle to determine the effect control parameters have on throughputs and downstream congestion levels. Results are developed without distributional assumptions on the packet arrival streams and make use of specially-tailored closed queueing networks. Performance comparisons between mechanisms are presented.

Risk-sensitive Estimation and A Differential Game

Ravi N. Banavar [*]   and Jason L. Speyer [†]

Abstract

A large deviation result is employed to solve the state estimation problem of a continuous time Gauss-Markov system with an exponential cost. The exponential cost is the expected value of an exponential function of the state estimation-error. A scalar θ appearing in the cost, termed as the risk factor, determines the penalty on the higher order moments of the error. In contrast to the minimum variance estimate, penalty on large deviations of the estimation error is possible.

1   Introduction

The expected value of an exponetial function as a performance measure - henceforth referred to as exponential cost (EC) - was initially proposed by Jacobson [11]. The cost is employed as an optimality criterion for a Gauss-Markov system with perfect knowledge of the systems states. The EC includes as a special case the LQG cost. The performance measure was later examined, for the case of imperfect state observation by [17, 20, 2, 12]. In [21] Whittle views the cost as a risk-sensitive criterion and lends a desirable certainty equivalence principle to the problem. An application to missile guidance is demonstrated in [18].

Interest in the exponential cost problem was revived when the solutions obtained

[*]Asst.Professor, Systems and Control Engineering, Indian Institute of Technology, Bombay - 400 076, INDIA.

[†]Professor, Mechanical, Aerospace and Nuclear Engineering Department, UCLA, Los Angeles, CA 90024

from it were linked to those of the H_∞ problem [8, 4]. In his initial paper, Jacobson [11] demonstrates the link between the EC and a differential game. Solutions to the H_∞ problem for time-varying systems is obtainable through a differential game approach [1, 15, 13]. The differential game commonality links the stochastic problem with the EC to the deterministic problem with a worst-case measure (H_∞). Most of the existing results on the exponential cost problem assume an exact equivalence to optimizing a quadratic performance measure. More recently, the problem has been re-examined from a large deviation perspective. Whittle [24, 25]. states a risk-sensitive maximum principle derived from a result in large deviation theory. Non-linear extensions to the same are found in [6]. Recasting the problem into a large deviation framework highlights the sub-optimal nature of the solution obtained through a quadratic kernel optimization. This is also in consonance with the deterministic H_∞ theory where sub-optimal solutions are obtained [4].

This paper examines the continuous time version of the estimation problem on a finite time interval. Speyer et al. [19] have considered the discrete time version of the problem. As stated in [19], the estimation problem in this stochastic setting is particularly interesting since the cost function is a curious exception to the family of functions proposed by Sherman [16, 9] that yield a conditional mean as the optimal estimator . Here, the performance measure is recast into a large deviation framework and a sub-optimal solution is obtained by invoking a result in large deviation. The approach is similar to Whittle's [25]. The two main theorems are stated and proved in section 2.

2 Preliminaries

Consider a Markov process

$$\dot{x} = Ax + Bw \quad x \in R^n, \quad w \in R^m \tag{1}$$

with a measurement

$$y = Cx + v \quad y \in R^p, \quad v \in R^p \tag{2}$$

where $f(x(t)|X(t-)) = f(x(t)|x(t-))$ is the Markov assumption and A, B, C are time-varying matrices of appropriate dimension. The system noise w and the measurement noise v are assumed to be white with zero mean and covariances W and V. The cost function is defined as

$$J \stackrel{\triangle}{=} E[e^{-\theta L}]$$
$$L \stackrel{\triangle}{=} \frac{1}{2} \int_0^T \| x - \hat{x} \|^2 \, d\tau \tag{3}$$

where $\| x - \hat{x} \|^2 \stackrel{\triangle}{=} (x - \hat{x})^T (x - \hat{x})$ and the expectation is over the random variables w, v and the initial state $x(0)$ and induced by the estimate $\hat{x}$. The cost function is minimized with respect to the estimate $\hat{x}(t)$

$$\min_{\hat{x}(.)} J \tag{4}$$

where $\hat{x}(t)$ is restricted to a causal function of the information $W(t)$ defined as

$$W(t) \stackrel{\triangle}{=} \{Y_t, \hat{X}_t, f(x(0))\}$$

where

$$Y_t \stackrel{\triangle}{=} \{y(\tau) : 0 \le \tau \le t\}$$

where

$$\hat{X}_t \stackrel{\triangle}{=} \{\hat{x}(\tau) : 0 \le \tau < t\}$$

and the distribution of the state at the initial time is Gaussian given by

$$f(x(0)) = \frac{1}{(2\pi)^{\frac{n}{2}} (det(P_0))^{0.5}} e^{(-\frac{1}{2}(x(0)-\hat{x}_0)^T P_0^{-1}(x(0)-\hat{x}_0)}$$

where $\hat{x}_0$ is the mean (a priori estimate at time $t = 0$) and $P_0 > 0$ is the covariance matrix.

The EC penalizes large deviations of the estimation error. A better picture of the cost function is obtained by expanding the exponential function as

$$J = E[1 - \theta L + \frac{\theta^2 L^2}{2!} - \frac{\theta^3 L^3}{3!} + \ldots] \tag{5}$$

As θ gets large ($>> 1$), the higher order moments $E[L^3], E[L^4], \ldots$ are weighted more heavily than $E[L], E[L^2]$. As θ gets small ($<< 1$), the lower order moments are weighted heavily and in particular, as $\theta \to 0$ the second moment is the dominant factor in the cost and the problem reduces to a minimum variance measure. The risk-factor θ allows a certain degree of freedom in shaping the probability density function of the estimation error.

3 Main Results

The problem is cast into a large-deviation [5] framework and the solution is obtained by invoking a large deviation result.

Preliminaries: Consider the course of the Markov process with state variables (x, z) where

$$\dot{x} = Ax + Bw$$

and

$$\dot{z} \stackrel{\Delta}{=} y = Cx + v$$

over the time interval $[0, T]$.

The derivative characteristic function H and the action functional D_{0T} [7] corresponding to this Markov process are defined as

$$H(x, \alpha, \beta) \stackrel{\Delta}{=} \alpha^T(Ax) + \beta^T(Cx) + \frac{1}{2}[\alpha^T W \alpha + \beta^T V \beta] \tag{6}$$

where α and β are conjugate variables (Lagrange Multipliers) and

$$D_{0T}(x(.), z(.)) \stackrel{\Delta}{=} \sup_{\alpha(.), \beta(.)} \int_0^T [\alpha^T \dot{x} + \beta^T \dot{z} - H(x, \alpha, \beta)] d\tau \tag{7}$$

where

$$z(.) \triangleq \{z(\tau) : 0 < \tau < T\}$$

$$x(.) \triangleq \{x(\tau) : 0 \le \tau < T\}$$

When the initial state $x(0)$ is unobserved and supposed random (here Gaussian) then

$$D_{0T}(x(.), z(.)) = D_0(x(0)) + \sup_{\alpha(.),\beta(.)} \int_0^T [\alpha^T \dot{x} + \beta^T \dot{z} - H(x, \alpha, \beta)] d\tau \qquad (8)$$

where

$$f(x(0)) \propto \exp^{-D_0(x(0))}$$

is the probability distribution of the initial state.

With these preliminaries in view, consider the cost

$$E[e^{-\theta L}]$$

It can be rewritten as

$$E[e^{-k\theta L}]$$

where k is a positive scalar parameter. Note that this modification alters the performance index but does not affect the optimal estimate. Let $\epsilon \triangleq \frac{1}{k}$. A large deviation result [5, 7] applied to the present problem states that

$$\lim_{\epsilon \to 0} \epsilon \log(E[e^{-\frac{1}{\epsilon}\theta L}]) = - \operatorname*{ess\,inf}_{x(.),z(.)} \{D_{0T}(x(.), z(.)) + \theta L\}$$

where "ess" is over the appropriate measure. From the above result, for large k the performance index

$$J = E[e^{-k\theta L}]$$

is logarithmically asymptotic

$$\exp\{-k \operatorname*{ess\,inf}_{x(.),z(.)} \{D_{0T}(x(.), z(.)) + \theta L\}\}$$

or

$$E[e^{-k\theta L}] \sim_L \exp\{-k \underset{x(.),\hat{x}(.)}{\text{ess inf}}\{D_{0T}(x(.), z(.)) + \theta L\}\} \tag{9}$$

where $\sim_L$ denotes logarithmic asymptoticity.

With this background, the main theorem of the paper is stated. The theorem below is akin to the risk sensitive maximum principle in [25] applied to the case of control with imperfect observation. Here it is derived for the problem of state estimation.

Theorem 1 The sub-optimal estimate $\hat{x}(t)$ is obtained by seeking the infimum of the linear-quadratic performance index

$$J_g = \frac{1}{\theta}D_0(x(0)) + \int_0^T \frac{1}{2} \parallel x - \hat{x} \parallel^2$$
$$+ \frac{1}{2\theta}[w^T W^{-1}w + (y - Cx)^T V^{-1}(y - Cx)]d\tau \tag{10}$$

with respect to the estimates $\{\hat{x}(\tau) : 0 \leq \tau < T\}$, and the supremum (infimum) with respect to $\{x(0), w(\tau) : 0 \leq \tau < T\}$ $\{y(\tau) : 0 < \tau < T\}$) for $\theta < 0$ ($\theta > 0$). The performance index is subject to (1).

Proof:

From (9)

$$J = E[e^{-k\theta L}]$$

$$\sim_L \exp\{-k \underset{x(.),\hat{x}(.)}{\inf}\{D_{0T}(x(.), z(.)) + \theta L\}\}$$

Let

$$J' \overset{\triangle}{=} \exp\{-k \underset{x(.),\hat{x}(.)}{\inf}\{D_{0T}(x(.), z(.)) + \theta L\}\} \tag{11}$$

Minimizing J with respect to $\hat{x}(.)$ is equivalent to minimizing J' with respect to $\hat{x}(.)$ We shall consider the case $\theta < 0$ only. The proof is similar for $\theta > 0$.

For $\theta < 0$

$$\underset{\hat{x}(.)}{\inf} J' \equiv \underset{\hat{x}(.)}{\inf} \exp\{-\theta k \underset{x(.),\hat{x}(.)}{\sup}\{\frac{1}{\theta}D_{0T}(x(.), z(.)) + L\} \tag{12}$$

$$\equiv \exp\{-\theta k \inf_{\hat{x}(.)} \sup_{z(.),\dot{z}(.)} \{\frac{1}{\theta} D_{0T}(x(.),z(.)) + L\}$$

$$\equiv \exp\{-\theta k \inf_{\hat{x}(.)} \sup_{x(0),w(.),y} \{\frac{1}{\theta} D_{0T}(x(.),z(.)) + L\} \tag{13}$$

Now examine the term

$$\sup_{\alpha(.),\beta(.)} \int_0^T [\alpha^T \dot{x} + \beta^T \dot{z} - H(x,\alpha,\beta)]d\tau$$

in the action functional (8). By a straightforward completion of squares it is shown to be

$$\int_0^T \frac{1}{2}[w^T W^{-1} w + v^T V^{-1} v]d\tau$$

which in turn can be expressed as

$$\int_0^T \frac{1}{2}[w^T W^{-1} w + (y - Cx)^T V^{-1}(y - Cx)]d\tau$$

and substituting in (8) results in

$$D_{0T}(x(.),z(.)) = D_0(x(0)) + \int_0^T \frac{1}{2}[w^T W^{-1} w + (y - Cx)^T V^{-1}(y - Cx)]d\tau \tag{14}$$

Define

$$J_g \triangleq \{\frac{1}{\theta} D_{0T}(x(.),z(.)) + L\} \tag{15}$$

From (13) and (14) the sub-optimal estimate is obtained by solving the differential game

$$\inf_{\hat{x}(.)} \sup_{x(0),w(.),y(.)} \{\frac{1}{\theta} D_{0T}(x(.),z(.)) + L\} \tag{16}$$

QED

□.

Remark 1:Note that in the case $\theta < 0$, the linear-quadratic problem is a differential game and for $\theta > 0$ the problem reduces to a one-sided optimization.

Remark 2: The estimate is termed *sub-optimal* since the the differential game formulation is equivalent to the original problem only in the limiting case as $\epsilon \to 0$ or $k \to \infty$.

Theorem 2 A saddle point solution to the differential game J_g exists if and only if there exists a solution to the Riccati differential equation

$$\dot{P} = PA^T + AP + BWB^T - P(C^TV^{-1}C + \theta I)P, \quad P(0) = P_0 \tag{17}$$

over the time interval $[0, T]$. The optimal values are then given by

$$x(0)^* = \hat{x}_0$$

$$w^* = WB^TP^{-1}(x - \hat{x})$$

$$y^* = C\hat{x}$$

$$\dot{\hat{x}} = A\hat{x} + PC^TV^{-1}(y - C\hat{x}); \quad \hat{x}(0) = \hat{x}_0 \tag{18}$$

Proof:

Assuming $J^*_{op}(x(.), \hat{x}(t), t)$ is the optimal return function [3] at time t. (All decisions are assumed optimal over the interval $[t, T]$ and $\hat{x}(t)$ is the a priori estimate at time t.) The saddle solution is sought by solving the Hamilton-Jacobi- Bellman partial differential equation [3] or Issacs' equation [10]

$$-\frac{\partial J^*_{op}}{\partial t} = \inf_{\hat{x}(t)} \sup_{x(t), w(t), y(t)} [\frac{\partial J^*_{op}}{\partial x}\dot{x} + M(x(t), \hat{x}(t), w(t), y(t), t)] \tag{19}$$

where

$$M(x(t), \hat{x}(t), w(t), y(t), t) \triangleq$$

$$\frac{1}{2} \parallel x(t) - \hat{x}(t) \parallel^2 + \frac{1}{2\theta}[w^T(t)W^{-1}w(t) + (y(t) - Cx(t))^TV^{-1}(y(t) - Cx(t))] \tag{20}$$

Assume a solution

$$J^*_{op}(x(.)(t), \hat{x}(t), t) = \frac{1}{2\theta}(x(.)(t) - \hat{x}(t))^TP^{-1}(t)(x(.)(t) - \hat{x}(t)) \tag{21}$$

with

$$J^*_{op}(x(.)(0), \hat{x}(0), 0) = \frac{1}{2\theta}(x(0) - \hat{x}(0))^TP^{-1}(0)(x(.)(0) - \hat{x}(0)) \tag{22}$$

Substituting (21) in (19) and subsequent algebraic manipulation results in

$$x(0)^{\bullet} = \hat{x}_0$$

$$w^{\bullet} = WB^T P^{-1}(x - \hat{x})$$

$$y^{\bullet} = C\hat{x}$$

$$\dot{\hat{x}} = A\hat{x} + PC^T V^{-1}(y - C\hat{x}); \quad \hat{x}(0) = \hat{x}_0 \tag{23}$$

where

$$\dot{P} = PA^T + AP + BWB^T - P(C^T V^{-1} C + \theta I)P \quad P(0) = P_0 \tag{24}$$

With the optimal values given by (23)

$$J_g(\hat{x}^{\bullet}, y, w, x(0)) \le J_g(\hat{x}^{\bullet}, y^{\bullet}, w^{\bullet}, x(0)^{\bullet}) \le J_g(\hat{x}, y^{\bullet}, w^{\bullet}, x(0)^{\bullet})$$

$$\forall x(0) \in R^n, \quad \hat{x}, x, y \in L_2[0, T]. \tag{25}$$

This proves sufficiency.

(Necessity):Assume that $P(t)$ becomes unbounded at time t_e, $0 \le t_e \le T$. $P(t) \to \infty$ as $t \to t_e$ from below. Let ϵ bea small number. The variation

$$\Delta J_g \overset{\triangle}{=} J_g(\hat{x}^{\bullet}, y, w, x(0)) - J_g(\hat{x}^{\bullet}, y^{\bullet}, w^{\bullet}, x(0))$$

$$=$$

$$\lim_{\to 0}\left(\frac{1}{2\theta} \| \Delta e^{\bullet}(t_e - \epsilon) \|^2_{P^{-1}(t_e - \epsilon)} + \frac{1}{2\theta} \int_0^{t_e - \epsilon} \| WB^T P^{-1}(\Delta e^{\bullet}) - w \|^2_{W^{-1}}\right.$$

$$+ \| y - C\hat{x}^{\bullet} \|^2_{V^{-1}})dt$$

$$+ \frac{1}{2} \int_{t_e - \epsilon}^{T} \| \Delta e^{\bullet} \|^2 + \frac{1}{\theta}(\| w \|^2_{W^{-1}} + \| y - Cx \|^2_{V^{-1}})dt \tag{26}$$

where $\Delta e^{\bullet} \overset{\triangle}{=} x - \hat{x}^{\bullet}$ represents the error in the optimal estimate. The existence of $P(t)$ is not assumed for the interval $[t_e - \epsilon, T]$ and consequently the optimal estimator given by (18) does not exist in this interval. In the interval $[t_e - \epsilon, T]$, $\hat{x}^{\bullet}$ represents

the optimal strategy of $\hat{x}$, no longer governed by the dynamics (18). Note that the optimal estimator over this interval of time may be non-linear as well.

Now consider the following strategies for y and w,

$$w = WB^T P^{-1} \Delta e^*, \quad y = C\hat{x}^* \quad \forall t \in [0, t_e - \epsilon]$$

$$w = 0, \quad y = Cx \quad \forall t \in [t_e - \epsilon, T] \tag{27}$$

Then

$$\Delta J_g = \lim_{\epsilon \to 0} \left(\frac{1}{2\theta} \parallel \Delta e^*(t_e - \epsilon) \parallel^2_{P^{-1}(t_e - \epsilon)} + \frac{1}{2} \int_{t_e - \epsilon}^{T} \parallel \Delta e^* \parallel^2 dt \right) \tag{28}$$

As $\epsilon \to 0$, $P^{-1}(t_e - \epsilon)$ tends to a singular matrix. For the interval, $[0, t_e - \epsilon]$, Δe^* is governed by

$$\begin{aligned} \Delta \dot{e}^* &= A\Delta e^* + B(WB^T P^{-1}\Delta e^*) \\ &= (A + BWB^T P^{-1})\Delta e^* \end{aligned} \tag{29}$$

The solution to the linear dynamic equation (29) can be expressed in terms of a state-transition matrix,

$$\Delta e^*(t_e - \epsilon) = \Phi((t_e - \epsilon), 0)\Delta e^*(0) \tag{30}$$

where $\Phi(.,.)$ is the state-transition matrix of $(A + BWB^T P^{-1})$ and $\Delta e^*(0) = x(0) - \hat{x}_0$. Since $x(0)$ is arbitrary, $x(0)$ can be chosen such that $\lim_{\epsilon \to 0} \Delta e^*(t_e - \epsilon)$ lies in the null space of $\lim_{\epsilon \to 0} P^{-1}(t_e - \epsilon)$. Note that the latter is a singular matrix. Then

$$\lim_{\epsilon \to 0} \left(\frac{1}{2\theta} \parallel \Delta e^*(t_e - \epsilon) \parallel^2_{P^{-1}(t_e - \epsilon)} = 0 \right) \tag{31}$$

and

$$\Delta J_g = \lim_{\epsilon \to 0} \frac{1}{2} \int_{t_e - \epsilon}^{T} \parallel \Delta e^* \parallel^2 dt > 0 \tag{32}$$

Therefore, (32) is a contradiction that $\hat{x}^*$ is optimal. Hence, $P(t)$ must exist $\forall t \in [0, T]$.

QED

□.

*Remark 3:*If the solution to the Riccati differential equation (17) exists over the time interval $[0, T]$ then the solution remains positive drefinite throughout the interval [15, 26].

Remark 4: For $\theta < 0$, the estimator views the initial state $x(0)$, the disturbance w and measuremnt corruption v as adversaries and adopts the safest (risk-averse) strategy.

For $\theta > 0$ the estimator views the actions of the initial state $x(0)$, the disturbance w and measuremnt corruption v as friendly and adopts a more adventurous (risk-prone) strategy.

Remark 5: Define an attenuation function

$$J_{af} \overset{\Delta}{=} \sup \frac{\int_0^T \| x - \hat{x} \|^2 \, dt}{\| x(0) - \hat{x}_0 \|^2_{P_0^{-1}} + \int_0^T \| w \|^2_{W^{-1}} + \| v \|^2_{V^{-1}} \, dt} \tag{33}$$

where $((x(0) - \hat{x}_0), w, v) \in (R^n, L_2[0, T], L_2[0, T])$ and sup stands for supremum. The sytem dynamics (1) and measurement (2) are now deterministic and the objective is to find an optimal estimator which bounds J_{af} i. e.

$$J_{af} < \frac{1}{-\theta}$$

wghere θ is a negative scalar. An optimal estimator for the performance measure (33) is identical to (18) and for the time invariant case (with a few additional assumptions) is the H_∞ optimal solution [14].

4 Conclusions

The large deviation approach applied to the exponential cost problem highlights the sub-optimal nature of the solution obtained through a quadratic kernel optimization. This feature is very much similar to that encountered in the deterministic counterpart or the H_∞ optimal problem. Even in the latter case, sub-optimal solutions are obtained. The estimator is not the typical conditional mean (Kalman)

filter and is an exception to the large class of cost functions proposed by Sherman. Existence of the optimal solution depends on the existence of the solution to a Riccati differential equation, which in turn depends on the risk-factor θ.

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CENTRALIZED AND DECENTRALIZED SOLUTIONS OF THE LINEAR-EXPONENTIAL-GAUSSIAN PROBLEM *

Chih-hai Fan,[†]   Jason L. Speyer,[‡]   and Christian R. Jaensch[§]

ABSTRACT

A particular class of stochastic control problems constrained to different information patterns is considered. This class consists of minimizing the expectation of an exponential cost criterion with quadratic argument subject to a discrete-time Gauss-Markov dynamic system, i.e., the Linear-Exponential-Gaussian (LEG) control problem. Besides the one-step delayed information pattern previously considered, the classical (includes current observations) and the one-step delayed information-sharing (OSDIS) patterns are assumed. After determining the centralized controller based upon the classical information pattern, the optimal decentralized controller based upon the OSDIS pattern and the solution to a static team problem is found to be affine. A unifying approach to determine controllers based upon these three information patterns is obtained by noting that the value of a quadratic exponent of an exponential function is independent of the information structure. Even though the controllers are determined by a backward recursion of this exponent, the value of the cost criterion is not; rather a coefficient of the exponential delineates the value of the cost criterion with respect to the information patterns. Both necessary and sufficient conditions for the controllers to be minimizing are obtained regardless of the exponential form. The negative exponential form is included which is unimodal but not convex.

1. INTRODUCTION

Control problems with an exponential cost, in particular the exponential of a quadratic form, have proved to be an interesting extension to control problems with only a quadratic cost, such as the well known linear-quadratic-gaussian (LQG) control problem. The first to consider the linear-exponential-gaussian (LEG problem was Jacobson [8] who treated the case of perfect state observation. Jacobson pointed out the relation between the LEG problem and deterministic differential games. Subsequently, Speyer, Deyst, and Jacobson in [18] derived results for special cases of the general LEG problem with imperfect observations where they obtained fixed finite-dimensional controllers. Speyer et. al. also considered the general LEG

problem with imperfect observations, but obtain an optimal controller which is a function of the entire smoothed history of the state vector from the initial to the current time. Kumar and Van Schuppen [13] obtained results analogous to [18]. In [21] Whittle obtained the solution to the general LEG problem with a one-step delayed information pattern and showed that a fixed-dimensional controller resulted. In his analysis, a risk-sensitive certainty equivalence principle is derived, from which the differential game interpretation follows. Later, Bensoussan and Van Schuppen obtained in [2] results in continuous time analogous to [21]. A major objective of this paper is to consider in a unified manner LEG problems having various information patterns. Included is the decentralized LEG problem which is more naturally formulated in a discrete-time setting. Therefore, we will not refer to the continuous time LEG optimal control problem any further.

In contrast to centralized control which assumes all the measurements of the system's state are available (*the classical information pattern*), decentralized control focuses on systems in which not all the information is available. A particular information structure is assumed here called the *one-step delayed information-sharing pattern* (OSDISP) which assumes that each control station (elements of the control vector) has, at the current time, all the previously implemented control values, all the observations made anywhere in the system through and including the previous time, and its own observation at the current time. Hence, the difference between the one-step delayed information-sharing pattern and the classical information pattern is that the current observations are not shared. The one-step delayed information pattern is that only the past observations are available and the current observations are not available to any control station.

Dynamic team problems associated with a one-step delayed information-sharing pattern can be conveniently reduced into a series of static team problem by utilizing the Dynamic Programming method. However, this is in general not true with arbitrary information pattern [24]. Nevertheless, for different cost functions the one-step delayed information-sharing pattern has produced recursive solutions for obtaining the optimal control law. In particular, Sandell and Athans developed in [17] the first recursive solution to the team problem with a quadratic cost, one-step delayed information-sharing pattern and linear dynamics. For the exponential of a quadratic cost function without intermediate state penalties, Krainak, Machell, Marcus and Speyer derived in [12] an analogous result for a one-step delayed information-sharing pattern. In section 6 we extend that result for the general exponential of a quadratic cost function.

In the following presentation superscripts on vectors indicate either components or partition of vectors. The following defines a class of static team problems.

Definition 1.1

A team decision problem is concerned with a decision making unit (called a team) consisting of M members, that chooses a value of the decision vector u from a subset $\Xi \subset R^p$ and incurs a cost $C(u,x)$, which depends upon both the decision u and the prevailing state of the world x. The decision variable u is usually a p-tuple of individual decision variables, or an M-tuple of individual decision vectors $u^i \in \Xi^i \subset R^{p_i}$, where

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$\sum_{i=1}^{M} p_i = p = \dim u$. The set of all possible decision values is some subset of the Cartesian product of the Ξ^i's, i.e. $\Xi \subset \Xi^1 \times \Xi^2 \times \cdots \times \Xi^M$. We also assume that $x \in \Omega$, where (Ω, Σ, P) is a given probability space and $\Omega \equiv R^n$ for some n, Σ consists of the Borel sets on R^n and P is a known probability measure. C(u,x) indicates the penalty associated with each decision for each state of the world and is assumed for all purposes of this paper to be real-valued and Borel measurable on $(R^p \times \Omega)$. Furthermore, the ith team member has available to him some observation of the state of the world : $z^i = h^i(x)$, where $\dim z^i = r^i$ and z is the vector of all the observation values known to the team members with $\dim z = r$. Note that z^i indicates the ith partition of z and not the ith component. It is also presumed that $h^i(\cdot) : \Omega \to R^{r^i}$ is a Borel measurable function to avoid the discussion of pathological cases. Finally, it is supposed that the set of admissible control laws for the ith team member, U^i, consists of all Borel measurable functions, $\gamma^i(\cdot) : R^{r^i} \to \Xi^i$, where Ξ^i is a Borel measurable subset of R^{p_i}. This means that the control value u^i is *solely* a function of the observation z^i. The set of all admissible team control laws is defined as $U_T = U^1 \times U^2 \times \cdots \times U^M$. The problem faced by the team is to select the control law

$$\gamma^* = \begin{bmatrix} \gamma^{*^1} \\ \vdots \\ \gamma^{*^M} \end{bmatrix} \in \begin{bmatrix} U^1 \\ \vdots \\ U^M \end{bmatrix}$$

which minimizes the average cost, $J = E[C(\gamma(z), x)]$. □

For the remaining part of this section and throughout the paper, we adhere to the notation given in the above definition, unless noted otherwise.

A sufficient condition for global optimality of a class of static team problems is given by Radner [15]. In particular, it is assumed that for every fixed $x \in R$, $C(u, x)$ is convex. Furthermore, because of the hard to verify "local finiteness" condition in [15], an extension which circumvents this conditions is given by Krainak et. al. in [11]. Further simplifications are obtained here because of the restriction to exponentials with quadratic arguments.

In section 2 and 3 we extend the results presented by Whittle in [21] for the slightly more natural assumption that the control at the current time is a function of the observation history up to the current time (the classical information pattern) rather than only up to the previous time (the one-step delayed information pattern). In addition, in section 2 and 3 we contribute detailed proofs to some of the Theorems of [21] where only a proof outline is given. The lemma given by Whittle in [21] converts the expectation operation of the exponential of the quadratic function to the extremization operation. Using this property, we avoid the tedious proof in [11] for the exponential cost and give sufficient conditions for both the risk-averse and risk-preferring cases of the LEG team problem in Section 4. The optimal team control function for risk-preferring is found also as an affine function. For the risk-preferring case, $C(u, x)$ is an exponential

function but nonconvex. This extends the results of [11] in which only the risk-averse case with a convex cost function is considered to be a nonconvex but unimodal exponential function. For more general functions of $C(u, x)$, the work in [11] must still be extended. In Section 5, we investigate the relationship among three different information patterns: the classical information, one-step delayed information, and one-step delayed information-sharing patterns. Using the results of the classical information and one-step delayed information patterns, the dynamic programming recursion will be shown to be essentially the same for the three different information patterns. This property simplifies the derivation of the control gains for the one-step delayed information-sharing pattern in that the backward algorithm in [10] is no longer needed. Furthermore, only the coefficient of the exponential in the cost function changes for different information patterns indicating their relative optimality. In section 6 we derive the optimal decentralized controllers for the one-step delayed information-sharing pattern. This extends the results of Krainak, Machell, Marcus and Speyer in [12], who examined only the case in which terminal state penalty is present. Also, the results in Section 6 are extended to nonconvex $C(u, x)$ but unimodal exponential functions.

2. THE LINEAR-EXPONENTIAL-GAUSSIAN PROBLEM

Consider the following linear, stochastic, discrete-time system,

$$x_{t+1} = A_t x_t + B_t u_t + w_t \tag{2.1}$$

where we assume that $\dim(x_t) = n$ and $\dim(u_t) = p$. The state observation is now restricted to the form

$$z_t = H_t x_t + v_t \tag{2.2}$$

where $z_t \in R^r$. Notice that the definition in (2.2) is different from that in [21]. In addition, x_0 is normally distributed with mean $\underline{x}_0$ and covariance $V_0 > 0$, and $\{w_t\}$ and $\{v_t\}$ are assumed to be zero-mean, jointly Gaussian, independent random variables for all $t = 0, 1, \cdots, N$ with known positive-definite covariance matrices W_t and Θ_t, respectively. Whereas this description of the dynamics suffices for the centralized LEG problem, a few additional requirements have to be added for the decentralized LEG problem.

Specifically, it is assumed that u_t and B_t are partitioned as

$$u_t = [(u_t^1)^T, \cdots, (u_t^M)^T]^T, \quad B_t = [B_t^1, B_t^2, \cdots, B_t^M] \tag{2.3}$$

where u_t^i corresponds to the control implemented at time t by the ith team member ($i \in \{1, \cdots, M\}$) with $\dim(u_t^i) = p_i$ and $\sum_{i=1}^M p_i = p$. In addition, z_t is partitioned as

$$z_t = [(z_t^1)^T, \cdots, (z_t^M)^T]^T \tag{2.4}$$

where z_t^i corresponds to the observation of the ith team member at time t with $\dim(z_t^i) = r_i$ and $\sum_{i=1}^M r_i = r$. It is also presumed that Θ_t can be partitioned as

$$\Theta_t = Diag[\Theta_t^1, \cdots, \Theta_t^M] \tag{2.5}$$

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In other words, v_t^i and v_t^j $(i, j \in \{1, \cdots, M\}, i \neq j)$ are assumed to be independent Gaussian random variables, where v_t^i corresponds to the noise corrupting the observation of the ith team member.

The objective is to minimize the following performance index,

$$J(\theta) = E[-\theta e^{-\frac{1}{2}\theta\Psi}] \tag{2.6}$$

where θ is a real scalar and Ψ is given by,

$$\Psi = \sum_{t=0}^{N-1} (x_t^T Q_t x_t + u_t^T R_t u_t) + x_N^T Q_N x_N \tag{2.7}$$

We assume that $Q_t \geq 0$ and $R_t > 0$, and R_t is partitioned in the decentralized problem as

$$R_t = \begin{bmatrix} R_t^{11} \cdots R_t^{1M} \\ \cdots\cdots\cdots \\ R_t^{M1} \cdots R_t^{MM} \end{bmatrix} \tag{2.8}$$

In addition, all matrices and vectors are assumed to be compatibly dimensioned.

The feature distinguishing the decentralized problem from the centralized is the assumed information pattern. In particular, a one-step delayed information-sharing pattern is assumed for the decentralized LEG problem which can be described as follows,

$$Y_0 \equiv \emptyset(empty\ set), \ Y_t \equiv Y_{t-1} \cup \{z_{t-1}, u_{t-1}\}, \ Y_t^i = Y_t \cup \{z_t^i\}$$

Given this, we require for the decentralized LEG control problem that the team member constructs his own control value, u_t^i, according to the mutually agreed law, $\gamma_t^i(\cdot) : Y_t^i \to R^{p_i}$, i.e. $u_t^i = \gamma_t^i(Y_t^i) \in U^i$, as discussed in Definition 1.1 where U_T is the admissible control set.

On the other hand, we require for the LEG problem with the classical information pattern that the control at time t be a Borel measurable function of the observation history $\tilde{Y}_t$, $\gamma_t(\cdot) : \tilde{Y}_t \to R^p$ i.e. $u_t = \gamma_t(\tilde{Y}_t)$, where $\tilde{Y}_0$ is the initial information available on x_0 and $\tilde{Y}_t = Y_t^1 \cup \cdots \cup Y_t^M$. The set of all admissible control functions with the classical information pattern is

$$U_C = \{u_t : u_t = \gamma_t(\tilde{Y}_t)\}$$

The information pattern based on $\tilde{Y}_t$ is called classical, the information based on Y_t^i is one-step delayed information-sharing, and the information pattern based on Y_t is called one-step delayed. Similar to the definition of U_C, the set of all admissible control functions for the one-step delayed information pattern is

$$U_S = \{u_t : u_t = \gamma_t(Y_t)\}$$

From the above definition, the following relation is obtained.

$$U_S \subset U_T \subset U_C \tag{2.9}$$

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In general, we will also adopt the notation,

$$X_t \equiv (x_0, x_1, \cdots, x_t), \quad X_t^N \equiv (x_t, x_{t+1}, \cdots, x_N)$$

with a similar convention for all other variables.

2.1 Some preliminaries

Two lemmas from Whittle [21] [22] are given here to form the basis for the dynamic programming methodology given in section 2.2. The first lemma is concerned with the relationship between minimization and integration of gaussian densities and is restated below somewhat more precisely.

Lemma 2.1 Let $S(u, v; \theta)$ be a quadratic form in the components of the vectors u and v with $\dim(v) = r$. In other words, let $\xi^T = [u^T, v^T]$, then

$$S(u, v; \theta) = \frac{1}{2}\xi^T \bar{S}\xi + k\xi + n \qquad \bar{S} = \left[\begin{array}{cc} S_{uu} & S_{uv} \\ S_{vu} & S_{vv} \end{array} \right]$$

If $\theta > 0$, suppose that $\bar{S} > 0$ and $S(u, v; \theta)$ attains its minimum at u $= u^*$ and v $= v^*$, whereas if $\theta < 0$, assume that $S_{uu} > 0, S_{vv} < 0$ and $S(u, v; \theta)$ attains its minimax solution at u $= u^*$ and v $= v^*$. Then

$$\min_u \int_{-\infty}^{\infty} -\theta e^{-\theta S(u,v;\theta)} dv = -\theta(2\pi)^{\frac{1}{2}r}|\theta S_{vv}|^{-\frac{1}{2}} e^{-\theta S(u^*,v^*;\theta)} \propto e^{-\theta S(u^*,v^*;\theta)} \tag{2.10}$$

and the minimum is attained at u $= u^*$.

Proof. See [9] [5] or [23]

The importance of this lemma is that it precisely relates the expectation operation on the exponential with quadratic argument with respect to a Gaussian probability density to the extremization with respect to the random variable of the quadratic argument of the exponential.

The second preliminary result concerns the explicit form of the conditional probability density of all unobservables and is stated in detail below. Let $f(X_N, Z_N | U_{N-1})$ denote the joint probability density function of the random variables X_N, Z_N given the admissible control sequence U_{N-1}, where

$$X_N \equiv (x_0, x_1, \cdots, x_N), \quad Z_N \equiv (z_0, z_1, \cdots, z_N), \quad U_{N-1} \equiv (u_0, u_1, \cdots, u_{N-1})$$

This notation is consistent with that of Whittle; see [23] equation (2.39).

Lemma 2.2:

$$f(X_N, Z_N | U_{N-1}) = \Pi_{k=1}^{N}[f(z_k | x_k) f(x_k | x_{k-1}, u_{k-1})] f(z_0 | x_0) f(x_0) \propto e^{-\frac{1}{2}(P + m_N)} \tag{2.11}$$

with

$$P = \sum_{k=0}^{N-1} (n_k + \sum_{i=1}^{M} (m_k^i)) + (x_0 - \underline{x}_0)^T V_0^{-1} (x_0 - \underline{x}_0)$$

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$$= \sum_{k=0}^{N-1} (n_k + m_k) + (x_0 - \underline{x}_0)^T V_0^{-1} (x_0 - \underline{x}_0)$$

$$n_k = (x_{k+1} - A_k x_k - B_k u_k)^T W_k^{-1} (x_{k+1} - A_k x_k - B_k u_k)$$

$$m_k^i = (z_k^i - H_k^i x_k)^T (\Theta_k^i)^{-1} (z_k^i - H_k^i x_k), \quad i \in \{1, \cdots, M\}, \quad m_k = (z_k - H_k x_k)^T (\Theta_k)^{-1} (z_k - H_k x_k)$$

and the constant term for (2.11) is

$$const_1 = (2\pi)^{-\frac{n}{2}} |V_0|^{-\frac{1}{2}} \Pi_{k=0}^{N-1} [(2\pi)^{-\frac{n}{2}} |W_k|^{-\frac{1}{2}}] \Pi_{k=0}^N [(2\pi)^{-\frac{r}{2}} |\Theta_k|^{-\frac{1}{2}}] \tag{2.12}$$

Proof. See [9] or [5].

Define

$$S_.^c = \Psi + \theta^{-1}(P + m_N) \tag{2.13}$$

S^c is called *"stress"* in [21] and Ψ is defined in (2.7). We shall also say that a variable extremizes S^c if it minimizes S^c in the case $\theta > 0$, and it maximizes S^c when $\theta < 0$.

2.2 The Dynamic Programming decomposition

The Dynamic Programming recursion for the classical information pattern is given as follows. The cost function is

$$J = \min_{U_{N-1}} E[-\theta e^{-\frac{1}{2}\theta \Psi}]$$

By applying the Fundamental Theorem of Dynamic Programming, see [18], J is written as follows,

$$J = E[\min_{u_0} E[\min_{u_1} E[\cdots \min_{u_t} E[\cdots \min_{u_{N-1}} E[E[-\theta e^{-\frac{1}{2}\theta \Psi} | \bar{Y}_N] | Y_N] \cdots | Y_{t+1}] \cdots | Y_2] | Y_1]]$$

The above equation can be rewritten more explicitly as follows,

$$J = \int_{-\infty}^{\infty} \min_{u_0} \int_{-\infty}^{\infty} \min_{u_1} \int_{-\infty}^{\infty} \cdots \min_{u_t} \int_{-\infty}^{\infty} [\cdots \min_{u_{N-1}} \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} -\theta e^{-\frac{1}{2}\theta \Psi}$$

$$f(X_N | \bar{Y}_N) dX_N] f(z_N | Y_N) dz_N \cdots f(z_{t+1} | Y_{t+1})] dz_{t+1} \cdots f(z_2 | Y_2) dz_2$$

$$f(z_1 | Y_1) dz_1 f(z_0) dz_0 = \int_{-\infty}^{\infty} \min_{u_0} [\int_{-\infty}^{\infty} \min_{u_1} [\int_{-\infty}^{\infty} \cdots \min_{u_t} [\int_{-\infty}^{\infty} \min_{U_{t+1}^{N-1}} [\int_{-\infty}^{\infty} -\theta e^{-\frac{1}{2}\theta \Psi}$$

$$f(X_N | \bar{Y}_N) f(Z_{t+2}^N | U_{N-1}, Z_{t+1}) dX_N dZ_{t+2}^N] f(Z_{t+1} | U_t) dz_{t+1}] \cdots dz_2] dz_1] dz_0 \tag{2.14}$$

From (2.14) the optimal return function is defined as,

$$J_{t+1}(\bar{Y}_{t+1}) = \min_{U_{t+1}^{N-1}} E[-\theta e^{-\frac{1}{2}\theta \Psi} | \bar{Y}_{t+1}] f(Z_{t+1} | U_t)$$

Then, the Dynamic Programming recursion rule is

$$J_t(\tilde{Y}_t) = \min_{u_t} \int_{-\infty}^{\infty} J_{t+1}(\tilde{Y}_{t+1}) dz_{t+1} \tag{2.15}$$

The optimal return function at the final stage, i.e. at t=N, after reviewing (2.14) is

$$J_N(\tilde{Y}_N) = (-\theta) f(\tilde{Y}_N | U_{N-1}) E[e^{-\frac{1}{2}\theta\Psi} | \tilde{Y}_N]$$

$$= (-\theta) \int_{-\infty}^{\infty} e^{-\frac{1}{2}\theta\Psi} f(X_N | \tilde{Y}_N) f(Z_N | U_{N-1}) dX_N$$

$$= (-\theta) \int_{-\infty}^{\infty} e^{-\frac{1}{2}\theta\Psi} f(X_N, Z_N | U_{N-1}) dX_N$$

From Lemma 2.2

$$J_N(\tilde{Y}_N) = const_1(-\theta) \int_{-\infty}^{\infty} e^{-\frac{1}{2}\theta S^c} dX_N \tag{2.16}$$

where $const_1$ is given in (2.12). Let $S^c_{X_N X_N}$ be equivalent to S_{vv} in Lemma 2.1 and apply Lemma 2.1 to (2.16). Let $\Phi_N(\tilde{Y}_N) = \text{ext}_{X_N} S^c$ where "ext" means "min" for $\theta > 0$ and "max" for $\theta < 0$, then $J_N(\tilde{Y}_n)$ can be written as

$$J_N(\tilde{Y}_N) = const_2(-\theta e^{-\frac{1}{2}\theta\Phi_N(\tilde{Y}_N)}), \quad const_2 = const_1(2\pi)^{\frac{n(N+1)}{2}} |\theta S^c_{X_N X_N}|^{-\frac{1}{2}}$$

where $\Phi_N(\tilde{Y}_N)$ is a quadratic function in Z_N and U_{N-1}.

The optimal return function at time stage N-1 is evaluated by using the Dynamic Programming recursion rule (2.15). Thus,

$$J_{N-1}(\tilde{Y}_{N-1}) = \min_{u_{N-1}} \int_{-\infty}^{\infty} J_N(\tilde{Y}_N) dz_N = \min_{u_{N-1}} \int_{-\infty}^{\infty} const_2(-\theta) e^{-\frac{1}{2}\theta\Phi_N(\tilde{Y}_N)} dz_N$$

Let $S^c_{z_N z_N}$ be equivalent S_{vv} in Lemma 2.1 and apply Lemma 2.1 to the above equation. Let $\Phi_{N-1}(\tilde{Y}_{N-1}) = \min_{u_{N-1}} \text{ext}_{z_N} \Phi_N(\tilde{Y}_N)$, then $J_{N-1}(\tilde{Y}_{N-1})$ can be written as

$$J_{N-1}(\tilde{Y}_{N-1}) = (const_3)(-\theta) e^{-\frac{1}{2}\theta\Phi_{N-1}(\tilde{Y}_{N-1})}, \quad const_3 = const_2(2\pi)^{\frac{r}{2}} |\theta S^c_{z_N z_N}|^{-\frac{1}{2}} \tag{2.17}$$

where $\Phi_{N-1}(\tilde{Y}_{N-1})$ is a quadratic function in Z_{N-1} and U_{N-2}.

In general, the recursion produces

$$J_t(\tilde{Y}_t) = const_4(-\theta) e^{-\frac{1}{2}\theta\Phi_t(\tilde{Y}_t)} \tag{2.18}$$

where $const_4 = const_3 \Pi_{k=t}^{N-1} [(2\pi)^{\frac{r}{2}} |\theta S^c_{z_k z_k}|^{-\frac{1}{2}}]$ and

$$\Phi_t(\tilde{Y}_t) = \min_{u_t} \text{ext}_{z_{t+1}} \Phi_{t+1}(\tilde{Y}_{t+1}) \tag{2.19}$$

$\Phi_t(\tilde{Y}_t)$ is a quadratic function in Z_t and U_{t-1}, and u_t found from (2.19) is the optimal control at time t for the classical information pattern.

3.THE OPTIMAL CONTROL LAW WITH THE CLASSICAL INFORMATION PATTERN

From the derivation in Section 2.2 the following Theorem is obtained, which is applied to the centralized LEG problem with current observation (the classical information pattern) and is different from Theorem 1 in Whittle [21] who considered the centralized LEG problem with the one-step delayed information pattern.

Theorem 3.1: Let S^c be positive definite in $U_t^{N-1}, X_N, Z_{t+1}^N$ when $\theta > 0$ and S^c be positive definite in U_t^{N-1} and negative definite in X_N, Z_{t+1}^N when $\theta < 0$. Suppose S^c is minimized with respect to the decision as yet undetermined $U_t^{N-1} = (u_t, \cdots, u_{N-1})$ and extremized with respect to X_N and the current unobservable Z_{t+1}^N for a given value of $\tilde{Y}_t$. Then, the value of u_t is the optimal control at time t and the order of the optimization is irrelevant.

Proof: From the Dynamic Programming decomposition in Section 2.2, the recursion $\Phi_t(\tilde{Y}_t)$ (2.19) for the classical information pattern is

$$\Phi_t(\tilde{Y}_t) = \min_{u_t} \operatorname*{ext}_{z_{t+1}} \Phi_{t+1}(\tilde{Y}_{t+1}) \tag{3.1}$$

$$\Phi_N(\tilde{Y}_N) = \operatorname*{ext}_{X_N} S^c(X_N, \tilde{Y}_N) \tag{3.2}$$

The optimal control at time t determined from (3.1) assumes the classical information pattern. $\Phi_t(\tilde{Y}_t)$ can be written as

$$\Phi_t(\tilde{Y}_t) = \min_{u_t}[\operatorname*{ext}_{z_{t+1}} \Phi_{t+1}(\tilde{Y}_{t+1})] = \min_{u_t}[\operatorname*{ext}_{z_{t+1}} \min_{u_{t+1}} \operatorname*{ext}_{z_{t+2}} \Phi_{t+2}(\tilde{Y}_{t+2})] = \min_{u_t}[\Phi_{t+2}(z_{t+1}^*, z_{t+2}^*, u_{t+1}^*, Z_t, U_t)] \tag{3.3}$$

By repeatedly using equation (3.1) in (3.3)

$$\Phi_t(\tilde{Y}_t) = \min_{u_t}[\Phi_N(Z_{t+1}^{N*}, Z_t, U_{t+1}^{N*}, U_t)] \tag{3.4}$$

From (3.2) and (3.4)

$$\Phi_t(\tilde{Y}_t) = \min_{u_t}[S^c(X_N^*, Z_{t+1}^{N*}, Z_t, U_{t+1}^{N*}, U_t)] = \min_{U_t^{N-1}} \operatorname*{ext}_{Z_{t+1}^N} \operatorname*{ext}_{X_N} S^c(X_N, \tilde{Y}_N) \tag{3.5}$$

Since we have assume S^c is positive definite in $U_t^{N-1}, X_N, Z_{t+1}^N$ when $\theta > 0$, so the order of the minimizations can be certainly interchanged. We also assume S^c is positive definite in U_t^{N-1} and negative definite in X_N, Z_{t+1}^N when $\theta < 0$, so S^c possesses a saddle point. The operations of min and max can commute [1]. $\square$

The recursion of the quadratic function S^c in (3.5) can be decomposed into a forward recursion P_t and a backward recursion F_t. In fact, the ability to decompose S^c in such a way is the key in solving the general centralized discrete time LEG problem because it allows for a separation of the control algorithm from the

estimation algorithm. In essence, this says that in some sense the separation principle [21] holds for this class of problems. Let

$$S_1(X_t, \bar{Y}_t) = \theta^{-1}(x_0 - \underline{x}_0)^T V_0^{-1}(x_0 - \underline{x}_0) + \sum_{k=0}^{t-1}(x_k^T Q_k x_k + u_k^T R_k u_k$$

$$+ \theta^{-1}[n_k(x_{k+1}, x_k, u_k) + m_k(z_k, x_k)]) + \theta^{-1}m_t(z_t, x_t) \tag{3.6}$$

$$S_2(X_t^N, Z_{t+1}^N, U_t^{N-1}) = \sum_{k=t}^{N-1}(x_k^T Q_k x_k + u_k^T R_k u_k$$

$$+ \theta^{-1}n_k(x_{k+1}, x_k, u_k)) + \theta^{-1}\sum_{k=t+1}^{N} m_k(z_k, x_k) + x_N^T Q_N x_N \tag{3.7}$$

where S_1 is named "*past stress*" and S_2 is named "*future stress*" in [21].

Since S_2 does not contain X_{t-1} and S_1 does not contain $U_t^{N-1}, X_{t+1}^N, Z_{t+1}^N$, and by Theorem 3.1 the order of the optimization is irrelevant. Decompose (3.5) as,

$$\min_{U_t^{N-1}} \ext_{Z_{t+1}^N} \ext_{X_N} S^c(X_N, \bar{Y}_N) = \ext_{x_t}[\ext_{X_{t-1}} S_1(X_t, \bar{Y}_t) + \min_{U_t^{N-1}} \ext_{X_{t+1}^N} \ext_{Z_{t+1}^N} S_2(X_t^N, Z_{t+1}^N, U_t^{N-1})] \tag{3.8}$$

Define P_t and F_t as follows,

$$P_t(x_t, \bar{Y}_t) \equiv \ext_{X_{t-1}} S_1(X_t, \bar{Y}_t) \tag{3.9}$$

$$F_t(x_t) \equiv \min_{U_t^{N-1}} \ext_{X_{t+1}^N} \ext_{Z_{t+1}^N} S_2(X_t^N, Z_{t+1}^N, U_t^{N-1}) \tag{3.10}$$

Also observe that the second summation term of S_2 simply vanishes when extremizing with respect to Z_{t+1}^N, since extremizing S_2 with respect to z_k for all $k \in \{t+1, t+2, ..., N\}$ yields $z_k^* = H_k x_k$ implying that $\sum m_k = 0$.

It also follows immediately from the above definitions of P_t and F_t that $P_t(x_t, \bar{Y}_t)$ satisfies the following *forward* recursion,

$$P_{t+1}(x_{t+1}, \bar{Y}_{t+1}) = \ext_{x_t}[P_t(x_t, \bar{Y}_t) + x_t^T Q_t x_t + u_t^T R_t u_t + \theta^{-1}(n_t + m_{t+1})]$$

with the initial condition,

$$P_0(x_0, \bar{Y}_0) = \theta^{-1}(x_0 - \underline{x}_0)^T V_0^{-1}(x_0 - \underline{x}_0) + \theta^{-1}m_0$$

and $F_t(x_t)$ satisfies the *backward* recursion,

$$F_t(x_t) = \min_{u_t} \ext_{x_{t+1}}[F_{t+1}(x_{t+1}) + x_t^T Q_t x_t + u_t^T R_t u_t + \theta^{-1}n_t] \tag{3.11}$$

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with the terminal condition $F_N(x_N) = x_N^T Q_N x_N$. From the above discussion, Theorem 3.2 is obtained which is an alternative form of Theorem 3.1. Theorem 3.2 assumes the centralized LEG problem with the classical information pattern and is different from Theorem 2 in [21] which considers the centralized LEG problem with the one-step delayed information pattern.

Theorem 3.2: Let $u^*(x_t, t)$ be the minimizing value of u_t in the recursion equation $F_t(x_t)$, and let $x_t^*(\bar{Y}_t)$ be the value of x_t extremizing $P_t(x_t, \bar{Y}_t) + F_t(x_t)$. Then, the optimal control at time t is given by,

$$u^*(x_t, t)|_{x_t=x_t^*} = u^*(x_t^*(\bar{Y}_t), t).$$

Proof: Since S_2 does not contain X_{t-1} and S_1 does not contain $U_t^{N-1}, X_{t+1}^N, Z_{t+1}^N$, and by Theorem 3.1 the order of the optimization is irrelevant, then from (3.8) to (3.10)

$$\min_{U_t^{N-1}} \operatorname*{ext}_{Z_{t+1}^N} \operatorname*{ext}_{X_N} S^c(X_N, \bar{Y}_N) = \operatorname*{ext}_{x_t}[\operatorname*{ext}_{X_{t-1}} S_1(X_t, \bar{Y}_t) + \min_{U_t^{N-1}} \operatorname*{ext}_{X_{t+1}^N} \operatorname*{ext}_{Z_{t+1}^N} S_2(X_t^N, Z_{t+1}^N, U_t^{N-1})] = \operatorname*{ext}_{x_t}[P_t + F_t]$$

If the state is completely observed, then the optimal control is $u_t^* = u^*(x_t, t)$ and is determined from the backward recursion $F_t(x_t)$ in (3.11). The optimal control in the case of imperfect state observation is obtained simply by replacing x_t by $x_t^*(\bar{Y}_t)$, where $x_t^*(\bar{Y}_t)$ is the value of x_t extremizing $P_t(x_t, \bar{Y}_t) + F_t(x_t)$. This replacement is a modified version of certainty equivalence [21]. $\square$

In the following, first the controller $u^*(x_t, t)$ is determined from the backward recursion $F_t(x_t)$ in Theorem 3.3, then, the forward recursion $P_t(x_t, \bar{Y}_t)$ is evaluated in Theorem 3.4A, and finally, the optimal controller at time t, $u^*(x_t^*(\bar{Y}_t), t)$, is computed in Theorem 3.5.

In fact, the following Theorems 3.3, 3.4 and 3.5 are taken from Whittle [21]. More detailed proofs than Whittle gave in [21] are given in [9]. The following Theorem essentially presents the results of solving the backward recursion $F_t(x_t)$.

Theorem 3.3: $F_t(x_t)$ is quadratic in the state variable x_t, $F_t(x_t) = x_t^T \Pi_t x_t$, and the optimal control for complete state information is linear, $u^*(x_t, t) = k_t x_t$, where the controller gain k_t satisfies

$$k_t = -R_t^{-1} B_t^T \Pi_{t+1}(I + B_t R_t^{-1} B_t^T \Pi_{t+1} + \theta W_t \Pi_{t+1})^{-1} A_t \qquad (3.12)$$

and Π_t satisfies the discrete time Riccati equation,

$$\Pi_t = Q_t + A_t^T \Pi_{t+1}(I + B_t R_t^{-1} B_t^T \Pi_{t+1} + \theta W_t \Pi_{t+1})^{-1} A_t, \quad \Pi_N = Q_N$$

For these assertion to be true when $\theta < 0$, it is necessary that $\Pi_{s+1} < -(\theta W_s)^{-1}$ for all $s \geq t$.

Proof: See [21] or [9]

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Remark: Notice Π_t is positive semidefinite when $\theta > 0$. From the backwards Riccati equation if $I + \theta W_t \Pi_{t+1} > 0$, then $\Pi \geq 0$ when $\theta < 0$.

The recursion equation for $P_t(x_t, \bar{Y}_t)$ is rewritten in the following form,

$$P_{t+1}(x_{t+1}, \bar{Y}_{t+1}) = \underset{x_t}{\text{ext}}[\bar{P}_t(x_t, Y_t) + x_t^T Q_t x_t + u_t^T R_t u_t$$

$$+ \theta^{-1}(n_t(x_{t+1}, x_t, u_t) + m_t(z_t, x_t))] + \theta^{-1} m_{t-1}(z_{t+1}, x_{t+1}) \tag{3.13}$$

where

$$\bar{P}_t(x_t, Y_t) = \underset{X_{t-1}}{\text{ext}}[\theta^{-1}(x_0 - \underline{x}_0)^T V_0^{-1}(x_0 - \underline{x}_0) + \sum_{k=0}^{t-1}(x_k^T Q_k x_k + u_k^T R_k u_k + \theta^{-1}(n_k + m_k))] \tag{3.14}$$

The reason for rewriting the P_t recursion as,

$$P_t(x_t, \bar{Y}_t) = \bar{P}_t(x_t, Y_t) + \theta^{-1} m_t(z_t, x_t) \tag{3.15}$$

is that the recursion equation for $\bar{P}_t$ is exactly the same as the equation for P_t in Whittle's paper. Therefore, we can invoke Theorem 4 in [21] to obtain an expression for $\bar{P}_t(x_t, Y_t) = P_t(x_t, \bar{Y}_t) - \theta^{-1} m_t(z_t, x_t)$ which will be needed in Theorem 3.5 to obtain the optimal controller with the classical information. A less complete form of this theorem without the explicit form of $L_{t+1}(Y_{t+1})$ (3.18) was originally given in [21]. However, as we shall see in Section 6 in order to obtain the optimal decentralized controller with the one-step delayed information-sharing pattern we need an explicit expression for $L_{t+1}(Y_{t+1})$ (3.18).

Theorem 3.4: $\bar{P}_t(x_t, Y_t)$ has the form

$$\bar{P}_t(x_t, Y_t) = \theta^{-1}(x_t - \bar{x}_t)^T V_t^{-1}(x_t - \bar{x}_t) + L_t(Y_t) \tag{3.16}$$

where $L_t(Y_t)$ contains all those terms independent of x_t and the matrix V_t and the vector $\bar{x}_t$ satisfy the recursions,

$$V_{t+1} = W_t + A_t(V_t^{-1} + H_t^T \Theta_t^{-1} H_t + \theta Q_t)^{-1} A_t^T \tag{3.17}$$

$$L_{t+1}(Y_{t+1}) = L_t(Y_t) + \bar{x}_t^T(\theta V_t)^{-1}\bar{x}_t + z_t^T(\theta \Theta_t)^{-1} z_t + u_t^T R_t u_t$$

$$- \theta^{-1}(V_t^{-1}\bar{x}_t + H_t^T \Theta_t^{-1} z_t)^T(V_t^{-1} + H_t^T \Theta_t^{-1} H_t + \theta Q_t)^{-1}(V_t^{-1}\bar{x}_t + H_t^T \Theta_t^{-1} z_t) \tag{3.18}$$

$$\bar{x}_{t+1} = A_t \bar{x}_t + B_t u_t + A_t(V_t^{-1} + H_t^T \Theta_t^{-1} H_t + \theta Q_t)^{-1}(H_t^T \Theta_t^{-1}(z_t - H_t \bar{x}_t) - \theta Q_t \bar{x}_t) \tag{3.19}$$

with the initial condition $\text{cov}(x_0) = V_0$, $L_0 = 0$, $\bar{x}_0 = \underline{x}_0$. Moreover, for these assertions to be true in the case $\theta < 0$, it is necessary that $V_s^{-1} + \theta Q_s + H_s^T \Theta_s^{-1} H_s > 0$ for all $s < t + 1$.

Proof: Proof of this theorem, especially (3.18) is given in Appendix A or in [9].

The following lemma will show the relationship of two inequalities that were used in the proof of Theorem 3.4 and the positive-definite property of V_t.

Lemma 3.1 The following inequality exists

$$V_t^{-1} + \theta Q_t + H_t^T \Theta_t^{-1} H_t > 0 \tag{3.20}$$

if and only if

$$V_t^{-1} + \theta Q_t + A_t^T W_t^{-1} A_t + H_t^T \Theta_t^{-1} H_t > 0 \tag{3.21}$$

for $\theta \neq 0$ where $t \in \{0, \cdots, N-1\}$. The existence of (3.20) implies that $V_{t+1} > 0$.

Proof: See Appendix B.

From Theorem 3.4, $\bar{x}_t$ is the value which optimizes $P_t(x_t, Y_t)$. The next theorem will find $\hat{x}_t$ which optimizes $P_t(x_t, \tilde{Y}_t)$.

Theorem 3.4A: $P_t(x_t, \tilde{Y}_t)$ has the form

$$P_t(x_t, \tilde{Y}_t) = \theta^{-1}(x_t - \hat{x}_t)^T \hat{V}_t^{-1}(x_t - \hat{x}_t) + \hat{L}_t(\tilde{Y}_t)$$

where $\hat{L}(\tilde{Y}_t)$ contains all those terms independent of x_t, and the matrix $\hat{V}_t$ and the vector $\hat{x}_t$ satisfy the recursion,

$$V_{t+1} = W_t + A_t(\hat{V}_t^{-1} + \theta Q_t)^{-1} A_t^T, \quad \hat{V}_{t+1} = (V_{t+1}^{-1} + H_{t+1}^T \Theta_{t+1}^{-1} H_{t+1})^{-1} \tag{3.22}$$

$$\hat{L}_t(\tilde{Y}_t) = L_t(Y_t) + \bar{x}_t[\theta V_t + (H_t^T(\theta \Theta_t)^{-1} H_t)^{-1}]^{-1} \bar{x}_t$$

$$+ z_t[\theta \Theta_t + H_t(\theta V_t) H_t^T]^{-1} z_t - 2\bar{x}_t[(\theta V_t)^{-1}(\theta \hat{V}_t) H_t(\theta \Theta_t)^{-1}] z_t$$

$$\hat{x}_t = \hat{V}_t(V_t^{-1}\bar{x}_t + H_t^T \Theta_t^{-1} z_t) = \bar{x}_t + \hat{V}_t H_t^T \Theta_t^{-1}(z_t - H_t \bar{x}_t) \tag{3.23}$$

$$\bar{x}_{t+1} = A_t(I + \theta \hat{V}_t Q_t)^{-1} \hat{x}_t + B_t u_t$$

with the initial condition $\hat{V}_0 = (V_0^{-1} + H_0^T \Theta_0^{-1} H_0)^{-1}$ and $\hat{x}_0 = (V_0^{-1} + H_0^T \Theta_0^{-1} H_0)^{-1}(V_0^{-1}\bar{x}_0 + H_0^T \Theta_0^{-1} z_0)$, where $V_t, \bar{x}_t$ are given in Theorem 3.4. Moreover, for these assertions to be true in the case $\theta < 0$, it is necessary that $\hat{V}_s^{-1} + \theta Q_s > 0$ for all $s < t+1$.

Proof: The proof is similar to that of Theorem 3.4 (See [5]).

Notice that the control gain k_t in (3.12) contains θW_t and the state estimate $\bar{x}_t$ (3.19) contains θQ_t. The

control gain depends on the covariance of the noise and the state estimate depends on the state-penalties. Nevertheless, the computation of the control law is decoupled from the computation of the state estimates implying that the separation principle holds even though not in the same strict sense as for the LQG problem.

We can now appeal to Theorem 3.2, 3.3, 3.4, 3.4A and the discussion leading to equations (3.13)-(3.15) to obtain the final result.

Theorem 3.5: The optimal control rule is $u^*(x_t^*, t) = k_t x_t^*$, where

$$x_t^* = (I + \theta V_t \Pi_t + V_t H_t^T \Theta_t^{-1} H_t)^{-1} (\bar{x}_t + V_t H_t^T \Theta_t^{-1} z_t) \tag{3.24}$$

$$= (I + \theta \hat{V}_t \Pi_t)^{-1} \hat{x}_t \tag{3.25}$$

and k_t, Π_t, V_t and $\bar{x}_t$ are given in Theorems 3.3 and 3.4 and $\hat{x}_t, \hat{V}_t$ are given in Theorem 3.4A. The necessary condition for (3.24) is $V_t^{-1} + \theta \Pi_t + H_t^T \Theta_t^{-1} H_t > 0$ when $\theta < 0$.

Proof:: As shown in Theorem 3.2 the optimal control is

$$u^*(x_t, t)|_{x_t = x_t^*} = u^*(x_t^*(\bar{Y}_t), t)$$

where x_t^* is determined by extremizing $P_t + F_t$ with respect to x_t,

$$\underset{x_t}{\text{ext}}[F_t + P_t] = \underset{x_t}{\text{ext}}[F_t + \bar{P}_t + \theta^{-1} m_t] \tag{3.26}$$

Using Theorem 3.3 and 3.4, the right hand side of the above equation can be written as,

$$\underset{x_t}{\text{ext}}[x_t^T \Pi_t x_t + \theta^{-1}(x_t - \bar{x}_t)^T V_t^{-1}(x_t - \bar{x}_t) + L_t(Y_t) + \theta^{-1}(z_t - H_t x_t)^T \Theta_t^{-1}(z_t - H_t x_t)] \tag{3.27}$$

Performing the indicated operation yields (3.24). Using Theorem 3.3 and 3.4A, the left hand side of (3.26) can be written as

$$\underset{x_t}{\text{ext}}[x_t^T \Pi_t x_t + \theta^{-1}(x_t - \hat{x}_t)^T \hat{V}_t^{-1}(x_t - \hat{x}_t) + \hat{L}_t(\bar{Y}_t)]$$

where the optimal value of x_t is given by (3.25). Therefore, x^* is obtained in (3.24) and (3.25), provided $(\theta V_t)^{-1} + \Pi_t + H_t^T(\theta \Theta_t)^{-1} H_t < 0$ when $\theta < 0$. Using (3.22)(3.23), it is easy to show that the two x_t^* are the same. Finally, from Theorem 3.2 $u^*(x_t^*, t) = k_t x_t^*$ and can thus deduce the Theorem. □

The form for the classical information pattern in (3.25) is similar to that of the one-step delayed information pattern in [21], only $\hat{V}_t, \hat{x}_t$ are changed to $\bar{V}_t, \bar{x}_t$. The form in (3.24) is an affine function which will be needed in Theorem 5.1.

Discussion of Sufficient and Necessary Conditions: The sufficient conditions given in Theorem 3.1 are different from the necessary conditions given in Theorems 3.3-3.5. Below we show that the necessity of Theorems 3.3-3.5 are also sufficient.

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Theorem 3.6: There exists an unique finite optimal control function which is given in Theorem 3.5 and yields finite cost, if and only if the following three inequalities exist for all $t \in \{0, \ldots, N-1\}$

1. $V_s^{-1} + \theta Q_s + H_s^T \Theta_s^{-1} H_s > 0$ for all $s < t$ when $\theta < 0$.

2. $V_t^{-1} + \theta \Pi_t + H_t^T \Theta_t^{-1} H_t > 0$ when $\theta < 0$.

3. $\theta W_s \Pi_{s+1} + I > 0$ for all $N > s \geq t$ when $\theta < 0$.

Proof: In order to obtain the optimal control function, F_t, P_t in theorems 3.3 and 3.4, and x_t^* (3.24) must exist. Condition 1 guarantees the existence of $\bar{P}_t$. Condition 2 is for the existence of x_t^* in (3.24). Condition 3 is from Theorem (3.3) and guarantees the existence of F_t in (3.11).

The necessary conditions are proved by contradiction. Notice S^c could be decoupled to four parts. The first part is from S_1(3.6) and (A.3)(see [9]), the second part is from Theorem 3.5, the third and fourth parts are from (3.7)(3.10) and (3.11).

$$
\begin{aligned}
S^c = & \sum_{k=0}^{t-1} (x_k - x_k^*)^T \theta^{-1} (V_k^{-1} + \theta Q_k + H_k^T \Theta_k^{-1} H_k + A_k^T W_k^{-1} A_k)(x_k - x_k^*) \\
& + (x_t - x_t^*)^T (\Pi_t + (\theta V_t)^{-1} + H_t^T (\theta \Theta_t)^{-1} H_t)(x_t - x_t^*) \\
& + \sum_{k=t}^{N-1} (x_{k+1} - x_{k+1}^*)^T [\Pi_{k+1} + (\theta W_k)^{-1}](x_{k+1} - x_{k+1}^*) \\
& + \sum_{k=t}^{N-1} (u_k - u_k^*)^T (R_k + B_k^T \Pi_{k+1} (\theta W_k \Pi_{k+1} + I)^{-1} B_k)(u_k - u_k^*)
\end{aligned}
$$

where

$$
x_k^*(k < t) = (V_k^{-1} + \theta Q_k + H_k^T \Theta_k^{-1} H_k + A_k^T W_k^{-1} A_k)^{-1}
$$

$$
\cdot (V_k^{-1} \bar{x}_k + H_k^T \Theta_k^{-1} z_k + A_k^T W_k^{-1} (x_{k+1} - B_k u_k))
$$

$$
x_t^* = (I + \theta \hat{V}_t \Pi_t)^{-1} \hat{x}_t
$$

$$
x_{k+1}^*(k \geq t) = (\theta \Pi_{k+1} + W_k^{-1})^{-1} W_k^{-1} (A_k x_k + B_k u_k)
$$

$$
u_k^*(k \geq t) = -(R_k + B_k^T \Pi_{k+1} (\theta W_k \Pi_{k+1} + I)^{-1} B_k)^{-1}
$$

$$
\cdot B_k^T \Pi_{k+1} (\theta W_k \Pi_{k+1} + I)^{-1} A_k x_k
$$

From Lemma 3.1 $V_k^{-1} + \theta Q_k + H_k^T \Theta_k^{-1} H_k + A_k^T W_k^{-1} A_k > 0$ if and only if $V_k^{-1} + \theta Q_k + H_k^T \Theta_k^{-1} H_k > 0$, therefore the form in condition 1 is obtained. Condition 2 is from the second part. Because condition 3 implies $\Pi_{k+1} \geq 0$ and $R_k + B_k^T \Pi_{k+1} (\theta W_k \Pi_{k+1} + I)^{-1} B_k > 0$, only condition 3 is needed for parts 3 and 4. If one of the three inequalities does not exist, then

$$
J = \min_{U_{N-1}} \int_{-\infty}^{\infty} (-\theta) e^{-\frac{1}{2} \theta S^c} dX_N dZ_N = \pm \infty
$$

Therefore, the expected value of the cost function would be infinite or negative infinite. The necessity of these conditions is established. □

Remark: Notice the three inequalities always exist when $\theta > 0$. Also, the three inequalities exist if and only if the assumption of S^c in Theorem 3.1 exists. Furthermore, the following relation associated with the gain (3.12) can be proved that

$$(R_t + B_t^T \Pi_{t+1}(\theta W_t \Pi_{t+1} + I)^{-1} B_t)^{-1} B_t^T \Pi_{t+1}(\theta W_t \Pi_{t+1} + I)^{-1} A_t$$
$$= R_t^{-1} B_t^T \Pi_{t+1}(I + B_t R_t^{-1} B_t^T \Pi_{t+1} + \theta W_t \Pi_{t+1})^{-1} A_t$$

Theorem 3.5 provides the discrete-time centralized controller with the classical information for finite horizon and time-varying coefficients. For the time-invariant, infinite horizon case under certain conditions, a time-invariant controller results. It is shown in [6] that this time-invariant controller is equivalent to a H_∞ controller, and the best H_∞ controller is produced when θ is decreased to a critical value θ_{cr} where the cost goes to infinity. However, the information pattern is not discussed. The controllers with the classical information or one-step delayed information patterns may be time-varying or time-invariant and finite horizon or infinite horizon. Therefore, the controllers here generalize H_∞ controller to a larger class of controllers.

4. DYNAMIC PROGRAMMING FOR THE DECENTRALIZED CONTROL PROBLEM

This section is begun by presenting a variation in the Dynamic Programming recursion given in Section 2.2. The difference is that instead of conditioning on the classical information pattern $\bar{Y}_t$ as in (2.18), the recursion is conditioned on the one-step delayed information pattern Y_t. This change allows the explicit formulation of the static team problem at each stage of the recursion. The analysis is the following. First, the Dynamic Programming recursion is obtained conditioned on Y_t. From the Dynamic Programming a recursion associated with the argument of the exponential $\Sigma_t(Y_t)$ is defined in (4.1). Note that $\Sigma_t(Y_t)$ and its recursion replace $\Phi_t(\bar{Y}_t)$ (2.18) and its recursion (2.19). $\Sigma_t(Y_t)$ is propagated backwards assuming that the decentralized controller is affine, which is shown to imply that it is quadratic at each stage time. In particular, in Section 4.1 it is shown that if Σ_{t+1} is quadratic, then the global optimal decentralized controller u_t^* is affine. Since $\Sigma_N(Y_N)$ is quadratic, then by induction $\Sigma_t(Y_t)$ will remain quadratic.

A deeper property of $\Sigma_t(Y_t)$, shown in Section 5.1, is that the quadratic form of $\Sigma_t(Y_t)$ at each time stage is independent of the information pattern. This is because the saddle point strategies for all the information patterns produce the same saddle point trajectory which is used to construct $\Sigma_t(Y_t)$. From this property a uniform approach is produced for the development of controllers with different information patterns; a considerable simplification over that given in [10].

The optimal return function is defined, rather than as in Section 2.2, as

$$J_{t+1}(Y_{t+1}) = \min_{U_{t+1}^{N-1}} E[-\theta e^{-\frac{1}{2}\theta \Psi} | Y_{t+1}] f(Z_t | U_t)$$

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Then, the Dynamic Programming recursion rule changes from $J_t(\tilde{Y}_t)$ in (2.15) to

$$J_t(Y_t) = \min_{u_t} \int_{-\infty}^{\infty} J_{t+1}(Y_{t+1}) dz_t = \int_{-\infty}^{\infty} J_{t+1}(\tilde{Y}_t) dz_t$$

In Lemma 5.1 the assumption that the controller is affine is used, and forms the basis for the propagation of the quadratic function Σ_t. The relationship between $J_t(Y_t)$ and $\Sigma_t(Y_t)$ is given as

$$J_t(Y_t) \propto \exp[-\frac{1}{2}\theta\Sigma_t(Y_t)], \quad \Sigma_t(Y_t) = \underset{z_t}{\text{ext}} \min_{u_t} \Sigma_{t+1}(Y_{t+1}) \tag{4.1}$$

where $\Sigma_{t+1}(Y_{t+1})$ is a quadratic function of Z_t and U_t, and $\Sigma_N =_{z_N}^{\text{ext}} \Phi_N(\tilde{Y}_N)$. From (2.19) and (4.1), the following relation is obtained.

$$\Sigma_t(Y_t) = \underset{z_t}{\text{ext}} \Phi_t(\tilde{Y}_t) \tag{4.2}$$

From (6.1) and (6.2), $\Sigma_{t+1}(Y_{t+1})$ can be decoupled as two parts

$$\Sigma_{t+1}(Y_{t+1}) = f_t(u_t, z_t, \bar{x}_t) + L_t(Y_t) \tag{4.3}$$

where $L_t(Y_t)$ is in (3.18) and $f_t(u_t, z_t, \bar{x}_t)$ will be given in (6.2). In the next section it is shown that the static team controller which globally minimizes f_t is affine. Therefore, starting with Σ_N, Σ_t by the recursion (4.1) is propagated backwards and by induction Σ_t remains quadratic.

4.1 The static team problem

To obtain the static optimal team controller we need a definition of person-by-person optimality. Then, it is shown that, with an additional assumption on the convexity of the quadratic function f_t, person-by-person optimality implies global team optimality. Although the cost C of Definition 1.1 is not explicitly defined, it is related to the optimal return function J_{t+1} as

$$\min_{u_t^i} E[C_t(\hat{u}_t(i, z_t, \bar{x}_t), z_t, \bar{x}_t, Y_t)|Y_t, z_t^i] = \min_{u_t^i} \int_{-\infty}^{\infty} J_{t+1}(Y_t, z_t, \hat{u}_t(i, z_t, \bar{x}_t)) d\hat{z}_t(i) \tag{4.4}$$

$$\propto \min_{u_t^i} \int_{-\infty}^{\infty} exp[-\frac{1}{2}\theta f_t(\hat{u}_t(i, z_t, \bar{x}_t), z_t, \bar{x}_t)] d\hat{z}_t(i) \propto exp[-\frac{1}{2}\theta \min_{u_t^i} \underset{\hat{z}_t(i)}{\text{ext}} f_t]$$

The last relation is from Lemma 2.1 and $\hat{z}_t(i) = \{z_t^j\}_{i \neq j}$, where $\hat{z}_t(i)$ is the vector z_t without the observation of the i-th team member, and

$$\hat{u}_t(i, z_t, \bar{x}_t) = \begin{cases} u_t^{j*}(z_t^j, \bar{x}_t) & if \quad i \neq j \\ u_t^i & if \quad i = j \end{cases}$$

Note that in [15] and [13] person-by-person optimality is defined with respect to the left side of (4.4). Below we define person-by-person optimality with respect to the right side of (4.4).

In the discussion of the static team problem, the subscript t is dropped for convenience.

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Definition 4.1:

A static team decision rule $u^* \in U_T$ is person-by-person optimal if for

$$J = \underset{z \in R^r}{\text{ext}} \, f(u, z, \bar{x}), \quad |J| < \infty$$

and if u^{i*} is determined from

$$\min_{u^i \in U_T^i} \, \underset{z(i)}{\text{ext}} \, f(\hat{u}(i, z \qquad \qquad \text{or } i \in \{1, 2, \cdots, M\}$$

□

Let f have the particular quadratic form

$$f(u, z, \bar{x}) = \bar{x}^T Q_{11} \bar{x} + 2\bar{x}^T Q_{12} z + 2\bar{x}^T Q_{13} u + z^T Q_{22} z + 2z^T Q_{23} u + u^T Q_{33} u \qquad (4.5)$$

where $\bar{x}$ is a constant vector and $Q_{11}, Q_{12}, Q_{13}, Q_{22}, Q_{23}$ and Q_{33} are defined in (6.3) and are function of θ, a real number and $\theta \neq 0$. The operator "ext" means "min" when $\theta > 0$ and "max" when $\theta < 0$. The vectors u and z are partitioned as $u = [u^{1T}, \cdots, u^{MT}]^T$ and $z = [z^{1T}, \cdots, z^{MT}]^T$, and the matrices $Q_{12}, Q_{13}, Q_{22}, Q_{23}$ and Q_{33} are partitioned into block form, to correspond to the partitioning of the vectors u and z, as

$$Q_{12} = [Q_{12}^1, \cdots, Q_{12}^M]; \quad Q_{13} = [Q_{13}^1, \cdots, Q_{13}^M]$$

$$Q_{\rho\rho} = \begin{bmatrix} Q_{\rho\rho}^{11} & \cdots & Q_{\rho\rho}^{1M} \\ \cdots & \cdots & \cdots \\ Q_{\rho\rho}^{M1} & \cdots & Q_{\rho\rho}^{MM} \end{bmatrix}; \quad \rho\rho \in \{22, 23, 33\}$$

Let us also denote,

$$\bar{R} = D^T Q_{33} D + Q_{23} D + (Q_{23} D)^T + Q_{22} \qquad (4.6)$$

$$\bar{\delta} = D^T Q_{33} C + (Q_{13} D)^T + Q_{23} C + Q_{12}^T \qquad (4.7)$$

where D and C are determined from (4.8) and (4.9).

It is shown in Lemma 4.1 that the person-by-person optimal team rule for the quadratic function f is affine. In Lemma 4.2, by completing the square and using a stronger assumption than required for person-by-person optimality, the affine person-by-person rule is also globally optimal.

Lemma 4.1: Let the quadratic function f be given as (4.5). If $Q_{33}^{ii} - (\beta^i)^T \alpha^i \beta^i > 0, \bar{R} > 0$ when $\theta > 0$, and if $Q_{33}^{ii} > 0$, $\bar{R} < 0$ when $\theta < 0$, then the optimal person-by-person controller for the one-step delayed information pattern is given by

$$u^* = C\bar{x} + Dz$$

where D is a block diagonal matrix with dimension $p_i \times r_i$ for the diagonal matrix D^i, and C is partitioned according to u into $p_i \times n$ blocks, $i \in \{1, \cdots, M\}$ where

$$D^i = -[Q_{33}^{ii} - \beta^{iT}\alpha^i\beta^i]^{-1}[(Q_{23}^{ii})^T - \beta^{iT}\alpha^i\gamma^i] \tag{4.8}$$

$$C^i = -[Q_{33}^{ii} - \beta^{iT}\alpha^i\beta^i]^{-1}[(Q_{13}^i)^T + \sum_{j=1,j\neq i}^{M} Q_{33}^{ij}C^j - \beta^{iT}\alpha^i\delta^i]_{j\neq i} \tag{4.9}$$

where for $i, j, k \in \{1, \cdots, M\}$,

α^i=ith minor of $D^T Q_{33} D + Q_{23} D + (Q_{23}D)^T + Q_{22}$: the ith minor does not depend on D^i,

$\beta^i = [Q_{23}^{ji} + D^{jT}Q_{33}^{ji}]_{j\neq i}$, $\qquad \gamma^i = [Q_{22}^{ji} + (Q_{23}^{ij}D^j)^T]_{j\neq i}$,

$\delta^i = [(Q_{12}^j + Q_{13}^j D^j)^T + \sum_{k=1,k\neq i}^{M} Q_{23}^{jk}C^k + \sum_{k=1,k\neq i}^{M} D^{jT}(Q_{33}^{jk}C^k)]_{j\neq i}$

Furthermore, the optimal cost is

$$\operatorname*{ext}_{z} f(u^*, z, \bar{x}) = f(u^*, z^*, \bar{x}) = \bar{x}^T(Q_{11} + Q_{13}C + (Q_{13}C)^T + C^T Q_{33} C - \bar{\delta}^T \bar{R}^{-1}\bar{\delta})\bar{x} \tag{4.10}$$

Proof: See Appendix C.

Remark: Note that $Q_{33}^{ii} = f_{u^i u^i}$, $\beta^i = f_{\hat{z}(i)u^i}$ and $(\alpha^i)^{-1} = f_{\hat{z}(i)\hat{z}(i)}$. For additional characterization of $\alpha^i, \beta^i, \gamma^i$ and δ^i, see equations from (C.3) to (C.6).

Remark: The optimal person-by-person control function is a stationary point for the function f.

Lemma 4.2: If $Q_{33} > 0$, the person-by-person optimal control function determined from Lemma 4.1 for the quadratic function $f(u, z, \bar{x})$ (4.5) with the one-step delayed information-sharing pattern (OSDISP) is also the unique global optimal team controller, i.e. for U_T defined in definition 1.1 and $\forall u \in U_T, u \neq u^*$, the inequality $f(u, z, \bar{x}) > f(u^*, z, \bar{x})$ holds.

Proof: From page 187 of [1], for a strictly convex function the person-by-person optimal solution is unique and is also team-optimal. In particular, since f is a quadratic function where it is assumed that $Q_{33} > 0$,

$$f(u^* + \delta u, z, \bar{x}) - f(u^*, z, \bar{x}) = \delta u^T Q_{33}\delta u + 2(Q_{33}(C\bar{x} + Dz) + Q_{23}^T z + Q_{13}^T \bar{x})^T \delta u$$

By applying $u^i = C^i\bar{x} + D^i z^i$ to $Q_{33}(C\bar{x} + Dz) + Q_{23}^T z + Q_{13}^T\bar{x}$, (C.1) is obtained. Since z satisfies the stationary conditions and δu^i depends on z^i only, $\hat{z}(i)$ in (C.1) is eliminated by using (C.2)

$$[Q_{33}(C\bar{x} + Dz) + Q_{23}^T z + Q_{13}^T\bar{x}]^T \delta u(z)$$

$$= \sum_{i=1}^{M}\{Q_{33}^{ii}u^i + [(Q_{13}^i)^T + \sum_{j=1,j\neq i}^{M} Q_{33}^{ij}C^j]\bar{x} + (Q_{23}^{ii})^T z^i + \beta^{iT}\hat{z}(i)\}^T \delta u^i(z)$$

$$= \sum_{i=1}^{M}(u^{i*} - C^i\bar{x} - D^i z^i)^T(Q_{33}^{ii} - \beta^{iT}\alpha^i\beta^i)\delta u^i(z^i) = 0$$

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Since $Q_{33} > 0$,

$$f(u^* + \delta u, z, \bar{x}) - f(u^*, z, \bar{x}) = \delta u^T Q_{33} \delta u > 0$$

Therefore, for arbitrary δu

$$f(u^* + \delta u, z, \bar{x}) > f(u^*, z, \bar{x})$$

□

The following Theorem shows that the optimal control $u^* \in U_T$ from Lemma 4.1 also minimizes $\int_{-\infty}^{\infty} -\theta e^{-\frac{1}{2}\theta f} dz$.

Theorem 4.1: Let the function f be a quadratic function given in (4.5) and $Q_{33} > 0$. Then, the control function $u^* \in U_T$ given in Lemma 4.1 also minimizes $-\theta e^{-\frac{1}{2}\theta f}$ and $\int_{-\infty}^{\infty} -\theta e^{-\frac{1}{2}\theta f} dz$.

Proof: Since $Q_{33} > 0$, f is a strictly convex function with respect to u and there exists an unique control function u^* which minimizes f. If $u \neq u^*$, $f(u^*, z, \bar{x}) < f(u, z, \bar{x})$. For $\theta \neq 0$, then

$$-\theta e^{-\frac{\theta}{2} f(u^*, z, \bar{x})} < -\theta e^{-\frac{\theta}{2} f(u, z, \bar{x})} \tag{4.11}$$

for $\forall u \in U_T, z$ and $\bar{x}$. From Theorem C in page 96 of [7] and equation (4.11), we know

$$\int_{-\infty}^{\infty} -\theta e^{-\frac{\theta}{2} f(u^*, z, \bar{x})} dz \leq \int_{-\infty}^{\infty} -\theta e^{-\frac{\theta}{2} f(u, z, \bar{x})} dz$$

If the equity exists, then

$$\int_{-\infty}^{\infty} -\theta [e^{-\frac{\theta}{2} f(u, z, \bar{x})} - e^{-\frac{\theta}{2} f(u^*, z, \bar{x})}] dz = 0$$

By using Theorem B on page 104 of [7]

$$-\theta e^{-\frac{\theta}{2} f(u^*, z, \bar{x})} = -\theta e^{-\frac{\theta}{2} f(u, z, \bar{x})}$$

The above equation contradicts equation (4.11). Therefore, only the inequality exists

$$\int_{-\infty}^{\infty} -\theta e^{-\frac{\theta}{2} f(u^*, z, \bar{x})} dz < \int_{-\infty}^{\infty} -\theta e^{-\frac{\theta}{2} f(u, z, \bar{x})} dz$$

for all $u \in U_T, z$, and $\bar{x}$. □

Remark: Theorem 4.1 gives the sufficient condition for the optimality of the team problem where $\theta \neq 0$. This extends the result in [13] which deals with the convex exponential function ($\theta < 0$) to a nonconvex but unimodal function ($\theta > 0$) as given in Theorem 4.1.

5. EFFECTS OF INFORMATION PATTERN ON DYNAMIC PROGRAMMING

RECURSION AND COST FUNCTION CRITERION

The main purposes of this section is to show that the recursion for Σ_t (4.1) is independent of the information pattern *and* that only the coefficient of the exponential $|\bar{R}|$ in (4.6) delineates the differences in the value of the cost.

From Theorem 3.5 and Theorem 4.1, the optimal control for the classical information and one-step delayed information-sharing patterns is an affine function. The optimal control for the one-step delayed information Pattern [21] can also be expressed as an affine function $u^* = C\bar{x} + Dz$, $D = [0]$. From the above discussion, the optimal control for the three different information patterns can be written as $u^* = C\bar{x} + Dz$, where C and D are different for different information patterns.

As will be seen, the variable z in Lemma 5.1 plays a role equivalent to v in Lemma 2.1. However, because $u^* = C\bar{x} + Dz$ is explicitly dependent on z, some modifications need to be done on Lemma 2.1 in which u, v are independent. Lemma 2.1 is a special case, i.e. $D = [0]$, of Lemma 5.1. Lemma 5.1 is needed for the Dynamic programming decomposition in Section 4 and gives the coefficient of the exponential $|\bar{R}|$.

Lemma 5.1: Let the function f defined in (4.5) be a strictly convex function with respect to u, i.e. $Q_{33} > 0$. Let u^* be the value which minimizes function f and $\int_{-\infty}^{\infty} -\theta e^{-\frac{\theta}{2}f(u,z,\bar{x})}dz$ and u^* is an affine function of z, i.e. $u^* = C\bar{x} + Dz$. If $\theta > 0$, assume $\bar{R} > 0$; if $\theta < 0$, assume $\bar{R} < 0$; and $\dim(z) = r$ where $\bar{R}$ is given in (4.6). Then,

$$\min_u \int_{-\infty}^{\infty} -\theta e^{-\frac{\theta}{2}f(u,z,\bar{x})}dz = -\theta(2\pi)^{\frac{1}{2}r}|\theta\bar{R}|^{-\frac{1}{2}}e^{-\frac{\theta}{2}f(u^*,z^*,\bar{x})} \propto e^{-\frac{\theta}{2}f(u^*,z^*,\bar{x})} \tag{5.1}$$

where z^* minimizes f when $\theta > 0$ and z^* maximizes f when $\theta < 0$.

Proof: Let the function f be defined as in (4.5) and u^* is the optimal u which minimizes f and $\int_{-\infty}^{\infty} -\theta e^{-\frac{\theta}{2}f(u,z,\bar{x})}dz$. The discussion at the beginning of section 5 indicates that the optimal control functions for the three different information patterns can be expressed as an affine function $u^* = C\bar{x} + Dz$. Notice that C and D are different for different information. Let $\bar{R}$ and $\bar{\delta}$ be defined in (4.6) and (4.7) and substitute $u^* = C\bar{x} + Dz$ into f of (4.5), then

$$f(u^*,z,\bar{x}) = z^T\bar{R}z + 2\bar{x}^T\bar{\delta}^T z + \bar{x}^T(C^TQ_{33}C + Q_{13}C + (Q_{13}C)^T + Q_{11})\bar{x}$$
$$= (z + \bar{R}^{-1}\bar{\delta}\bar{x})^T\bar{R}(z + \bar{R}^{-1}\bar{\delta}\bar{x}) + \bar{x}^T(Q_{\bar{x}\bar{x}} - \bar{\delta}^T\bar{R}^{-1}\bar{\delta})\bar{x}$$

where $Q_{\bar{x}\bar{x}} = C^TQ_{33}C + Q_{13}C + (Q_{13}C)^T + Q_{11}$. Also define $\bar{z} = z + \bar{R}^{-1}\bar{\delta}\bar{x}$, then

$$\int_{-\infty}^{\infty} -\theta e^{-\frac{\theta}{2}f(u^*,z,\bar{x})}dz = \int_{-\infty}^{\infty} -\theta e^{-\frac{\theta}{2}\bar{z}^T\bar{R}\bar{z}}d\bar{z}e^{-\frac{\theta}{2}\bar{x}^T(Q_{\bar{x}\bar{x}} - \bar{\delta}^T\bar{R}^{-1}\bar{\delta})\bar{x}} \tag{5.2}$$

$$= -\theta(2\Pi)^{\frac{r}{2}}|\theta\bar{R}|^{-\frac{1}{2}}e^{-\frac{\theta}{2}\bar{x}^T(Q_{\bar{x}\bar{x}} - \bar{\delta}^T\bar{R}^{-1}\bar{\delta})\bar{x}} = -\theta(2\Pi)^{\frac{r}{2}}|\theta\bar{R}|^{-\frac{1}{2}}e^{-\frac{\theta}{2}\min_u^{\text{minext}}\,z\,f} \tag{5.3}$$

In order to obtain (5.3) from (5.2) we need to assume $\bar{R} > 0$ when $\theta > 0$ and $\bar{R} < 0$ when $\theta < 0$. If the assumption is not satisfied, the integration of (5.2) will be infinite ($\theta < 0$) or negative infinite ($\theta > 0$). From Theorem 4.1 and (5.3),

$$\min_u \int_{-\infty}^{\infty} -\theta e^{-\frac{\theta}{2}f(u,z)}dz = \int_{-\infty}^{\infty} -\theta e^{-\frac{\theta}{2}f(u^*,z,\bar{x})}dz \propto e^{-\frac{\theta}{2}\min_u^{\text{minext}}\,z\,f}$$

□

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5.1 The recursion of $\Sigma_t(Y_t)$ and its independence of the information pattern

In this section we give a theorem which shows that the argument Σ_t is the same for the classical information, one-step delayed information and one-step delayed information-sharing patterns (OSDISP). This will simplify the derivation about the control gains for OSDISP, i.e. the backward iterative algorithm given in [10] is no longer needed.

Theorem 5.1: Let S^c be positive definite in U_t^{N-1}, X_N, Z_t^N when $\theta > 0$ and S^c be positive definite in U_t^{N-1} and negative definite in X_N, Z_t^N when $\theta < 0$. Then, the value of $\Sigma_t(Y_t)$ in (4.1) is the same for the three different information patterns: the classical information, one-step delayed information, and one-step delayed information-sharing patterns.

Proof. In Section 4, the recursion from the Dynamic Programming decomposition produces (4.1) and (4.2) where $\Phi_t(\tilde{Y}_t)$ is given in (3.5). The order of the extremization in (3.5) is irrelevant (Theorem 3.1). By using Theorem 3.1, and the operations and definitions used in Section 3, an expression for Σ_t is obtained for the classical information pattern as

$$
\begin{aligned}
\Sigma_t(Y_t) &= \operatorname*{ext}_{x_t} \Phi_t(\tilde{Y}_t) = \min_{U_t^{N-1}} \operatorname*{ext}_{Z_t^N} \operatorname*{ext}_{X_N} S^c(X_N, \tilde{Y}_N) \\
&= \operatorname*{ext}_{x_t} \operatorname*{ext}_{x_t}[\operatorname*{ext}_{X_{t-1}} S_1(X_t, \tilde{Y}_t) + \min_{U_t^{N-1}} \operatorname*{ext}_{X_{t+1}^N} \operatorname*{ext}_{Z_{t+1}^N} S_2(X_t^N, Z_{t+1}^N, U_t^{N-1})] \\
&= \operatorname*{ext}_{x_t} \operatorname*{ext}_{x_t}[P_t + F_t] = \operatorname*{ext}_{x_t} \operatorname*{ext}_{x_t}[\bar{P}_t + \theta^{-1} m_t + F_t] \\
&= \operatorname*{ext}_{x_t} \operatorname*{ext}_{x_t}[\theta^{-1}(x_t - \bar{x}_t)^T V_t^{-1}(x_t - \bar{x}_t) + L_t(Y_t) + \theta^{-1}(z_t - H_t x_t)^T \Theta_t^{-1}(z_t - H_t x_t) + x_t^T \Pi_t x_t] \quad (5.4)
\end{aligned}
$$

where S_1, S_2, F_t, P_t are defined in (3.6)-(3.10). If $(\theta V_t)^{-1} + \Pi_t + H_t^T(\theta\Theta_t)^{-1}H_t < 0$ when $\theta < 0$ and clearly $(\theta V_t)^{-1} + \Pi_t + H_t^T(\theta\Theta_t)^{-1}H_t$ is always positive definite when $\theta > 0$, then the stationary condition of x_t which optimizes (5.4) is (3.24). Substitute x_t^* into (5.4)

$$
\operatorname*{ext}_{z_t}\{z_t^T[\theta\Theta_t + H_t((\theta V_t)^{-1} + \Pi_t)^{-1}H_t^T]^{-1}z - 2\bar{x}_t^T(\theta V_t)^{-1}[(\theta V_t)^{-1} + \Pi_t + H_t^T(\theta\Theta_t)^{-1}H_t]^{-1}H_t^T(\theta\Theta_t)^{-1}z_t
$$

$$
+ \bar{x}_t^T[\theta V_t + (\Pi_t + H_t^T(\theta\Theta_t)^{-1}H_t)^{-1}]^{-1}\bar{x}_t + L_t(Y_t)\} \tag{5.5}
$$

Assume $\theta\Theta_t + H_t((\theta V_t)^{-1} + \Pi_t)^{-1}H_t^T < 0$ when $\theta < 0$ and clearly $\theta\Theta_t + H_t((\theta V_t)^{-1} + \Pi_t)^{-1}H_t^T$ is always positive definite when $\theta > 0$. Then, from the first-order stationary condition of z_t for optimizing (5.5), we obtain

$$
z_t^* = H_t(I + \theta V_t \Pi_t)^{-1}\bar{x}_t \tag{5.6}
$$

If z_t^* is substituted into (5.5), then we obtain (See Appendix D for details of derivation)

$$
\Sigma_t = \bar{x}_t^T \Pi_t[\theta V_t \Pi_t + I]^{-1}\bar{x}_t + L_t(Y_t) \tag{5.7}
$$

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For the one-step delayed information pattern, Z_t^N is not available at time t. From the assumption about S^c, the order of the optimizaton is irrelevant w.r.t. U_t^{N-1}, Z_t^N, X_N.

$$\min_{U_t^{N-1}} \text{ext}_{Z_t^N} \text{ext}_{X_N} S^c(X_N, \tilde{Y}_N) = \text{ext}_{z_t}[\bar{P}_t + \text{ext}_{z_t} \theta^{-1} m_t + F_t]$$

$$= \text{ext}_{z_t}[x_t^T \Pi_t x_t + \theta^{-1}(x_t - \bar{x}_t)^T V_t^{-1}(x_t - \bar{x}_t) + L_t(Y_t)] \tag{5.8}$$

Extremizing with respect to z_t will make $m_t = 0$. If $\Pi_t + (\theta V_t)^{-1} < 0$ when $\theta < 0$ where clearly, $\Pi_t + (\theta V_t)^{-1}$ is always positive definite when $\theta > 0$, the optimal x_t^* will be as follows.

$$x_t^* = (I + \theta V_t \Pi_t)^{-1} \bar{x}_t \tag{5.9}$$

By substituting x_t^* in (5.9) into (5.8), (5.7) is obtained. Therefore, the classical information pattern and the one-step delayed information pattern have the same Σ_t. In order to obtain x_t^* in (3.24) and (5.9), and z_t^* in (5.6), we assume $(\theta V_t)^{-1} + \Pi_t + H_t^T(\theta \Theta_t)^{-1} H_t < 0$, $\Pi_t + (\theta V_t)^{-1} < 0$, $\theta \Theta_t + H_t((\theta V_t)^{-1} + \Pi_t)^{-1} H_t^T < 0$ when $\theta < 0$. The assumption that S^c is positive definite in U_t^{N-1} and negative definite in X_N, Z_t^N when $\theta < 0$ will guarantee the three inequalities exist. This assumption on S^c guarantees that $J_t(Y_t)$ in (4.2) is finite. If any one of the three inequalities is not satisfied, then $J_t(Y_t)$ will be infinite.

From (2.9), $U_S \subset U_T \subset U_C$. If g is a strictly convex function with respect to u, then

$$\min_{u \in U_S} g \geq \min_{u \in U_T} g \geq \min_{u \in U_C} g$$

If $\min_{u \in U_S} g = \min_{u \in U_C} g$, then $\min_{u \in U_S} g = \min_{u \in U_T} g = \min_{u \in U_C} g$. We write in a more explicit form at time stage $N - 1$

$$\Sigma_{N-1} = \text{ext}_{z_{N-1}} \min_{u_{N-1}} \Sigma_N = \min_{u_{N-1}} \text{ext}_{Z_{N-1}^N} \text{ext}_{X_N} S^c$$

Because $U_S \subset U_T \subset U_C$

$$\min_{u_{N-1} \in U_S} \text{ext}_{Z_{N-1}^N} \text{ext}_{X_N} S^c \geq \min_{u_{N-1} \in U_T} \text{ext}_{Z_{N-1}^N} \text{ext}_{X_N} S^c \geq \min_{u_{N-1} \in U_C} \text{ext}_{Z_{N-1}^N} \text{ext}_{X_N} S^c$$

Σ_t has shown to be the same for the classical and one-step delayed information patterns, therefore

$$\min_{u_{N-1} \in U_S} \text{ext}_{Z_{N-1}^N} \text{ext}_{X_N} S^c = \min_{u_{N-1} \in U_C} \text{ext}_{Z_{N-1}^N} \text{ext}_{X_N} S^c$$

From the above two equations

$$\min_{u_{N-1} \in U_S} \text{ext}_{Z_{N-1}^N} \text{ext}_{X_N} S^c = \min_{u_{N-1} \in U_T} \text{ext}_{Z_{N-1}^N} \text{ext}_{X_N} S^c = \min_{u_{N-1} \in U_C} \text{ext}_{Z_{N-1}^N} \text{ext}_{X_N} S^c$$

At $t = N - 1$ the classical information, one-step delayed information and one-step delayed information-sharing patterns have the same Σ_{N-1}

$$\Sigma_{N-1}(Y_{N-1}) = \bar{x}_{N-1}^T \Pi_{N-1}[\theta V_{N-1} \Pi_{N-1} + I]^{-1} \bar{x}_{N-1} + L_{N-1}(Y_{N-1})$$

By induction, using the recursion for Σ_t, the value of Σ_t for the decentralized case with the one-step delayed information-sharing pattern is giving by (5.7). □

Remark: From Theorem 5.1 and (4.3), the value of $f_t(u_t^*, z_t^*, \bar{x})$ are the same for the three information patterns. Therefore, from (4.10) and (5.7), $\bar{x}_t^T \Pi_t [\theta V_t \Pi_t + I]^{-1} \bar{x}_t = \bar{x}_t^T (Q_{11,t} + Q_{13,t} C_t + (Q_{13,t} C_t)^T + C_t^T Q_{33,t} C_t - \bar{\delta}_t^T \bar{R}_t^{-1} \bar{\delta}_t) \bar{x}$.

Remark: $\bar{R}$ defined in (4.6) is different for different information patterns because D is different for different information patterns. In Theorem 5.1 we showed that the value of Σ_t is the same for the three information patterns. Therefore, only $|\bar{R}|$ in (5.1) makes the cost function different for different information patterns.

5.2 Comparison of two solutions using the classical information pattern

In Speyer, Deyst, and Jacobson [18] the optimal controller for the classical information pattern is derived as a function of the entire smoothed history of the state vector from initial to the current time. This controller appears to be different from that given in Theorem 3.5. Two reasons for the difference are : 1) as shown below the probability density functions are the same, they are functionally expressed differently, and 2) the order of the extremization (or integration) is different. The probability density function given in Lemma 2.2 is

$$f(X_N, Z_N | U_{N-1}) = \Pi_{k=1}^N [f(z_k | x_k) f(x_k | x_{k-1}, u_{k-1})] f(z_0 | x_0) f(x_0)$$
$$\propto \exp\{\sum_{k=0}^{N-1} n_k + \sum_{k=0}^N m_k + (x_0 - \bar{x}_0)^T V_0^{-1} (x_0 - \bar{x}_0)\}$$

The probability density function which is used in [18] is

$$f(X_N, Z_N | U_{N-1}) = f(X_N | Z_N, U_{N-1}) \Pi_{i=1}^{N-1} f(z_{i+1} | U_i) f(z_0)$$

From Bayes theory, the above two equations are equivalent.

$$f(X_N | Z_N, U_{N-1}) \Pi_{i=1}^{N-1} f(z_{i+1} | U_i) f(z_0) = \Pi_{k=1}^N [f(z_k | x_k) f(x_k | x_{k-1}, u_{k-1})] f(z_0 | x_0) f(x_0)$$

Therefore, the exponent in [18] is equivalent to S^c defined in (2.13). From Lemma 2.1 the integration operation is equivalent to the extremization operation and from Theorem 3.1 the order of the extremization is irrelevant in deriving the optimal control function. Therefore, below we show that the value of the control given by the controller in Theorem 3.5 and [18] are equivalent and the values of the performance index are equal.

The optimal control function is given in Theorem 3.5. By substituting $\bar{x}_i$ (3.19) and u_{i-1}^*, $i \in \{1, \cdots, t\}$, into u_t^*, we will obtain

$$u_t^* = u_t^*(Z_t, \bar{x}_0) = \sum_{i=0}^t \Omega_i z_i + \Gamma \bar{x}_0 \tag{5.10}$$

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where $\hat{\Omega}_i$ and Γ depend on $D_i, i \in \{0, \cdots, t\}$. From Theorem 4.1 u_i^* is unique, i.e. D_i in (4.8) is unique. Therefore, Ω_i and Γ are unique, and in a similar way the controller u_i^* in [18] can be reduced to (5.10). Thus, substituting $u_i^* i \in \{0, \cdots, t\}$ into $\int J_{t+1}(\tilde{Y}_{t+1}) dz_{t+1}$, where $J_{t+1}(\tilde{Y}_{t+1})$ is in (2.18), and integrating w.r.t. Z_{t+1} will produce the same value of the performance index.

6. DERIVATION OF THE OPTIMAL DECENTRALIZED CONTROL LAW

A recursion based on (5.7) is developed in this section to determine the decentralized control function with one-step delayed information-sharing pattern. From (4.1) and (5.7), then

$$\Sigma_t = \underset{z_t}{\text{ext}} \min_{u_t \in U_T} [\bar{x}_{t+1}^T \hat{\Pi}_{t+1} \bar{x}_{t+1} + L_{t+1}(Y_{t+1})] \tag{6.1}$$

where $\hat{\Pi}_{t+1} = \Pi_{t+1}[\theta V_{t+1} \Pi_{t+1} + I]^{-1}$, and $L_{t+1}(Y_{t+1})$ and $\bar{x}_{t+1}$ are given by the forward recursion equations (3.18) and (3.19), respectively. Substitute $L_{t+1}(Y_{t+1})$ and $\bar{x}_{t+1}$ into (6.1), and after considerable simplifications Σ_t will be (See [9] for the simplification)

$$\Sigma_t = \underset{z_t}{\text{ext}} \min_{u_t \in U_T} \{\bar{x}_t^T[(\theta V_t)^{-1} + (\theta V_t)^{-1}(\Lambda_t A_t^T \hat{\Pi}_{t+1} A_t \Lambda_t - \Lambda_t)(\theta V_t)^{-1}]\bar{x}_t + 2\bar{x}_t^T(\theta V_t)^{-1}(\Lambda_t A_t^T \hat{\Pi}_{t+1} A_t \Lambda_t$$
$$- \Lambda_t)H_t^T(\theta \Theta_t)^{-1} z_t + 2\bar{x}_t^T(\theta V_t)^{-1}\Lambda_t A_t^T \hat{\Pi}_{t+1} B_t u_t + 2z_t^T(\theta \Theta_t)^{-1} H_t \Lambda_t A_t^T \hat{\Pi}_{t+1} B_t u_t$$
$$+ z_t^T[(\theta \Theta_t)^{-1} + (\theta \Theta_t)^{-1} H_t(\Lambda_t A_t^T \hat{\Pi}_{t+1} A_t \Lambda_t - \Lambda_t)H_t^T(\theta \Theta_t)^{-1}]z_t + u_t^T[R_t + B_t^T \hat{\Pi}_{t+1} B_t]u_t + L_t(Y_t)\}$$
$$= \underset{z_t}{\text{ext}} \min_{u_t \in U_T} [f_t + L_t(Y_t)] \tag{6.2}$$

where

$$f_t = \bar{x}_t^T Q_{11,t} \bar{x}_t + 2\bar{x}_t^T Q_{12,t} z_t + 2\bar{x}_t^T Q_{13,t} u_t + z_t^T Q_{22,t} z_t + 2z_t^T Q_{23,t} u_t + u_t^T Q_{33,t} u_t$$

$$Q_{11,t} = (\theta V_t)^{-1} + (\theta V_t)^{-1}[\Lambda_t A_t^T \hat{\Pi}_{t+1} A_t \Lambda_t - \Lambda_t](\theta V_t)^{-1}$$

$$Q_{12,t} = (\theta V_t)^{-1}[\Lambda_t A_t^T \hat{\Pi}_{t+1} A_t \Lambda_t - \Lambda_t]H_t^T(\theta \Theta_t)^{-1}, \quad Q_{13,t} = (\theta V_t)^{-1}\Lambda_t A_t^T \hat{\Pi}_{t+1} B_t$$

$$Q_{22,t} = (\theta \Theta_t)^{-1} + (\theta \Theta_t)^{-1} H_t[\Lambda_t A_t^T \hat{\Pi}_{t+1} A_t \Lambda_t - \Lambda_t]H_t^T(\theta \Theta_t)^{-1} \tag{6.3}$$

$$Q_{23,t} = (\theta \Theta_t)^{-1} H_t \Lambda_t A_t^T \hat{\Pi}_{t+1} B_t, \quad Q_{33,t} = R_t + B_t^T \hat{\Pi}_{t+1} B_t$$

and $\hat{\Pi}_{t+1}$ and Λ_t are defined as,

$$\hat{\Pi}_{t+1} = \Pi_{t+1}[\theta V_{t+1} \Pi_{t+1} + I]^{-1}, \quad \Lambda_t = [(\theta V_t)^{-1} + Q_t + H_t^T(\theta \Theta_t)^{-1} H_t]^{-1} \tag{6.4}$$

Notice the value of $\hat{\Pi}_{t+1}$ is invariant for the three information patterns.

6.1 The decentralized controller

Notice that since $L_t(Y_t)$ in (6.2) is not a function of z_t and u_t, and it would only act as an added constant to the performance index in Lemma 4.2. Applying Lemma 4.2 to (6.2) yields the optimal decision gain

equations at time stage t. In particular, the optimal decentralized controller at stage time t equals,

$$
u_t = \begin{bmatrix} D_t^1 \cdots 0 \\ \cdots \cdots \\ 0 \cdots D_t^M \end{bmatrix} \begin{bmatrix} z_t^1 \\ \vdots \\ z_t^M \end{bmatrix} + \begin{bmatrix} C_t^1 \\ \vdots \\ C_t^M \end{bmatrix} \bar{x} \tag{6.5}
$$

and the optimal decision gains D_t^i and C_t^i are determined from the following coupled matrix equations,

$$
D_t^i = -[R_t^{ii} + B_t^{iT}\hat{\Pi}_{t+1}B_t^i - (\beta_t^i)^T\alpha_t^i\beta_t^i]^{-1}[B_t^{iT}\hat{\Pi}_{t+1}A_t\Lambda_t(H_t^i)^T(\theta\Theta_t^i)^{-1} - (\beta_t^i)^T\alpha_t^i\gamma_t^i] \tag{6.6}
$$

$$
C_t^i = -[R_t^{ii} + B_t^{iT}\hat{\Pi}_{t+1}B_t^i - (\beta_t^i)^T\alpha_t^i\beta_t^i]^{-1}[B_t^{iT}\hat{\Pi}_{t+1}A_t\Lambda_t(\theta V_t)^{-1} + \sum_{j=1, j\neq i}^{M} Q_{33,t}^{ij}C_t^j - (\beta_t^i)^T\alpha_t^i\delta_t^i] \tag{6.7}
$$

where R^{ii} is the diagonal matrix defined in (2.8), B_t^i is given in (2.3), β_t^i and α_t^i are defined in section 4 using (6.3), and $i, j \in \{1, \cdots, M\}$.

Using the uniqueness of the optimal controller for the one-step delayed information pattern, a simpler form of C_t for the one-step delayed information-sharing pattern is given in the following lemma.

Lemma 6.1: A simpler form of the coefficient matrix C_t for the one-step delayed information-sharing pattern is

$$
C_t = (k_t - D_t H_t)(I + \theta V_t \Pi_t)^{-1} \tag{6.8}
$$

where D_t is given in (6.6). Furthermore, z_t^* is the same for the three information patterns : the classical, one-step delayed, and one-step delayed information-sharing patterns.

Proof: The notation (.) is used to distinguish the coefficients C_t, D_t and u_t^*, z_t^* for different information patterns. Therefore, (1), (2), and (3) denote the one-step delayed, the one-step delayed information-sharing, and an alternative form of the one-step delayed information patterns, respectively. The optimal u_t and z_t for the one-step delayed information pattern [21] are

$$
u_t^*(1) = C_t(1)\bar{x}_t + D_t(1)z_t = k_t(I + \theta V_t\Pi_t)^{-1}\bar{x}_t
$$

$$
z_t^*(1) = H_t(I + \theta V_t\Pi_t)^{-1}\bar{x}_t \tag{6.9}
$$

where $D_t(1)$ is a zero matrix and

$$
C_t(1) = k_t(I + \theta V_t\Pi_t)^{-1} \tag{6.10}
$$

For the one-step delayed information-sharing pattern

$$
u_t^*(2) = C_t(2)\bar{x}_t + D_t(2)z_t = (C_t(2) - D_t(2)\hat{R}_t^{-1}\delta_t)\bar{x}_t + D_t(2)(z_t + \hat{R}_t^{-1}\delta_t\bar{x}_t) \tag{6.11}
$$

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$$z_t^*(2) = -\bar{R}_t^{-1}\delta_t\bar{x}_t \qquad (6.12)$$

where $C_t(2), D_t(2)$ are given in (6.5)-(6.7), and $\bar{R}_t, \delta_t$ are given in (4.6)(4.7). The equation (6.11) will be needed to derive a simpler form of $C_t(2)$. From Theorem 5.1, the exponent recursion $\Sigma_t(Y_t)$ is the same for the three information patterns: the one-step delayed, classical, and one-step delayed information-sharing patterns. From (5.7)

$$\Sigma_t(Y_t) = \bar{x}_t^T\Pi_t[\theta V_t\Pi_t + I]^{-1}\bar{x}_t + L_t(Y_t) = f_t(u_t^*, z_t^*, \bar{x}_t) + L_t(Y_t)$$

where $f_t(u_t, z_t)$ is defined in (6.2) and $L_t(Y_t)$ is a constant term when Y_t is given. Therefore, $f_t(u_t^*, z_t^*, \bar{x}_t)$ is the same for the three different information patterns.

$$f_t(u_t^*(1), z_t^*(1), \bar{x}_t) = f_t(u_t^*(2), z_t^*(2), \bar{x}_t) = \bar{x}_t^T\Pi_t[\theta V_t\Pi_t + I]^{-1}\bar{x}_t \qquad (6.13)$$

When we consider $f_t(u_t^*, z_t^*)$, z_t^* in (6.12) will eliminate the second term in (6.11). Through observing (6.11) and (6.12), we can define $z_t^*(3), u_t^*(3)$ as in (6.15)-(6.17) and obtain the following relation

$$f_t(u_t^*(2), z_t^*(2), \bar{x}_t) = f_t(u_t^*(3), z_t^*(3), \bar{x}_t) \qquad (6.14)$$

$$u_t^*(3) = C_t(3)\bar{x}_t + D_t(3)z_t = (C_t(2) - D_t(2)\bar{R}_t^{-1}\delta_t)\bar{x}_t \qquad (6.15)$$

$$z_t^*(3) = -\bar{R}_t^{-1}\delta_t\bar{x}_t \qquad (6.16)$$

$$D_t(3) = [0], \quad C_t(3) = C_t(2) - D_t(2)\bar{R}_t^{-1}\delta_t \qquad (6.17)$$

From (6.13)(6.14)

$$f_t(u_t^*(1), z_t^*(1), \bar{x}_t) = f_t(u_t^*(3), z_t^*(3), \bar{x}_t) = \bar{x}_t^T\Pi_t[\theta V_t\Pi_t + I]^{-1}\bar{x}_t \qquad (6.18)$$

From (6.18), $u_t^*(3), z_t^*(3)$ is also the optimal value for the one-step delayed information pattern. The corollary 4.5 in [1] and Theorem 1.11 in [20] give sufficient conditions of the unique optimal value for a strictly convex-concave function and a strictly convex function, respectively. Since S^c is strictly convex with respect to X_N, U_t^{N-1}, Z_t^N when $\theta > 0$ and S^c is strictly concave with respect to X_N, Z_t^N and strictly convex w.r.t. U_t^{N-1} when $\theta < 0$, then the sufficient conditions for the unique optimal value are satisfied. Therefore, the optimal value of u_t, z_t for the one-step delayed information pattern is unique, i.e. $u_t^*(3) = u_t^*(1)$ and $z_t^*(3) = z_t^*(1)$. From (6.9) and (6.16)

$$-\bar{R}_t^{-1}\delta_t\bar{x}_t = H_t(I + \theta V_t\Pi_t)^{-1}\bar{x}_t \qquad (6.19)$$

From (6.10)(6.17) and $u_t^*(3) = u_t^*(1)$

$$C_t(3) = C_t(1) = k_t(I + \theta V_t \Pi_t)^{-1} = C_t(2) - D_t(2)\bar{R}_t^{-1}\delta_t$$

From the above two equations, the following simplified equation for C_t follows as

$$C_t(2) = D_t(2)\bar{R}_t^{-1}\delta_t + C_t(3) = D_t(2)\bar{R}_t^{-1}\delta_t + k_t(I + \theta V_t \Pi_t)^{-1} = (k_t - D_t(2)H_t)(I + \theta V_t \Pi_t)^{-1} \quad (6.20)$$

From (6.9), (6.19) and (5.6), z_t^* is the same for the three different information patterns : the one-step delayed, one-step delayed information-sharing and classical information patterns.□

By using (6.8), u_t^* with OSDISP is written as

$$u_t^* = k_t(I + \theta V_t \Pi_t)^{-1}\bar{x}_t + D_t(z_t - H_t(I + \theta V_t \Pi_t)^{-1}\bar{x}_t) \quad (6.21)$$

. The above form also exists for the classical and one-step delayed information patterns, where $D_t = 0$ for the one-step delayed information pattern and $D_t = k_t(I + \theta V_t \Pi_t + V_t H_t^T \Theta_t^{-1} H_t)^{-1} V_t H_t^T \Theta_t^{-1}$ for the classical information pattern. Since $z_t^* = H_t(I + \theta V_t \Pi_t)^{-1}\bar{x}_t$ will eliminate the second term of (6.21), only the first term influences the value of $f_t(u^*, z_t^*, \bar{x}_t)$. Therefore, the value of f_t and $\Sigma_t(Y_t)$ are the same for the three information patterns. □

Remark: Note that if C_t is chosen as in (6.8) with arbitrary D_t, then $f_t(u_t^*, z_t^*, \bar{x}_t)$ and $\Sigma_t(Y_t)$ still retain the same values.

The necessary and sufficient conditions for the decentralized controller with the one-step delayed information-sharing pattern are listed in following theorem.

Theorem 6.1: Consider the dynamic LEGT one-step delayed information-sharing problem specified by (2.1)-(2.8). There is an unique finite optimal team control law at time t given as

$$\gamma_t^i = D_t^i z_t^i + C_t^i \bar{x}_t \quad i \in \{1, \cdots, M\} \quad (6.22)$$

which yields finite cost, if and only if the following conditions are satisfied,

1. $V_s^{-1} + \theta Q_s + H_s^T \Theta_s^{-1} H_s > 0$ for all $s < t + 1$ when $\theta < 0$
2. $\theta \Pi_{s+1} + W_s^{-1} > 0$ for all $N > s \geq t + 1$ when $\theta < 0$.
3. $V_{t+1}^{-1} + \theta \Pi_{t+1} > 0$ when $\theta < 0$
4a. $\bar{R}_t = D_t^T Q_{33,t} D_t + Q_{23,t} D_t + (Q_{23,t} D_t)^T + Q_{22,t} < 0$ when $\theta < 0$
4b. $\bar{R}_t > 0$ and $Q_{33,t}^{ii} - (\beta_t^i)^T \alpha_t^i \beta_t^i > 0$ when $\theta > 0$

where the estimate of the state based upon the one-step delayed information pattern $\bar{x}_t$ is given by equation (3.19), the optimal gain matrices D_t^i and C_t^i, for $i \in \{1, \cdots, M\}$ are determined from the coupled matrix

equations (6.6) and (6.7), or the simplified form of C_t in (6.20). In addition, the matrix $\hat{\Pi}_t$ is determined by (6.4) .

Proof: In order to obtain the decentralized controller at time t, $\bar{P}_{t+1}, L_{t+1}, F_{t+1}$ in Theorem 3.3 and 3.4, x^*_{t+1} in (3.24) and (5.9), z^*_{t+1} in (5.6), D^i_t in (6.6) and C^i_t in (6.7) need exist. Condition 1 guarantees the existence of $\bar{P}_{t+1}, L_{t+1}$ (see Appendix A). Condition 2 is from (3.11) and guarantees the existence of F_{t+1}. Condition 3 guarantees that x^*_{t+1} in (3.24) and (5.9), and z^*_{t+1} in (5.6) exist (see Section 5.1). From Lemma 4.1, condition 4 and $Q_{33,t} > 0$ will guarantee the existence of D^i_t and C^i_t. Condition 2 implies $\Pi_{t+1} \geq 0$. Notice that the assumption $R_t > 0$, condition 3 and $\Pi_{t+1} \geq 0$ imply $Q_{33,t} > 0$. We prove necessity as following. If any one of conditions 1-4 does not exist, then from (2.10) and (5.2) the expected value of the cost will become infinite, regardless of the choice of the control function. This implies the necessity of conditions 1-4. □

Observe that the conditions 1-4 in the above Theorem are not contradictory, and since we are working with a discrete time system there is only a finite number of the conditions. Hence, there exists an $\epsilon > 0$, such that the conditions 1-4 are satisfied for all $|\theta| < \epsilon$. Therefore, D^i_t and C^i_t can be determined through Theorem 6.1.

7. CONCLUSION

The results presented here extend known optimal solutions of centralized and decentralized control problems with an exponential cost criterion, and in addition set forth an innovative methodology for the solution of optimal team problems with gaussian noise processes and exponential cost function. In Sections 2 and 3 we discussed the optimal solution of the centralized LEG problem under the hypothesis that the controller is constructed from the past information and the current observation of state. This extends the results of Whittle [21] who only considered a control law based upon the past information. It is shown that many of the prevalent concepts associated with the LQG problem, such as the optimality of linear Markov controllers, Riccati equations and the separation principle, hold for this class of problems in a somewhat modified form. Lemmas 4.1, 4.2 and Theorem 4.1 give sufficient conditions for global optimality of the LEG static team problem for $\theta \neq 0$. For $\theta > 0$, the exponential of the quadratic function is nonconvex but unimodal. This extends the result of Krainak et. al. [11] who only consider the convex exponential function ($\theta < 0$). Using the optimal control for the classical information and the one-step delayed information patterns, we prove in Theorem 5.1 that the value of the argument of the exponential is the same for different information patterns. Therefore, only $|\bar{R}|$ in the coefficient of the exponential cost function changes for different information. Through Theorem 5.1, an innovative method of deriving the optimal control gains for the team problem is given. The backward iterative process in [10] is no longer needed. In Section 6 we examined the complete dynamic LEGT problem with the one-step delayed information-sharing pattern for $\theta \neq 0$, and derived a set

of coupled algebraic equations satisfied by the optimal decision gains at each time stage. This extends the results of Krainak et. al. [12] who examined cost criterion in which only the terminal state penalties are present. The results in Section 6 is also applied to the nonconvex exponential function with the quadratic function ($\theta > 0$), where in [12] only convex exponential function ($\theta < 0$) is considered.

Given the continuous results of [4] and [16] the centralized and decentralized synthesis results here, given certain restrictions, generalized current H_∞ results to time-varying and time-invariant discrete-time systems over finite and infinite time horizons. Furthermore, in the derivations of the LEG controller a quadratic cost is constructed which is minimized with respect to the control variables, but maximized with respect to the input uncertainties when $\theta < 0$. This indicate a relationship between the LEG problem and game theory.

APPENDICES

A

Proof of Theorem 3.4.

Proof of (3.17) and (3.19) follows Whittle's paper. The derivation of $L_t(Y_t)$ (3.18) can be found in [9]. However, a simpler derivation of $L_t(Y_t)$ will be given in this section. The forward recursion (3.14) is as follows.

$$\bar{P}_{t+1}(x_{t+1}, Y_{t+1}) = \underset{x_t}{\text{ext}}[\bar{P}_t(x_t, Y_t) + x_t^T Q_t x_t + u_t^T R_t u_t + \theta^{-1}(n_t(x_{t+1}, x_t, t) + m_t(z_t, x_t))] \quad \text{(A.1)}$$

If the relation $\bar{P}_t(x_t, Y_t) = (x_t - \bar{x}_t)^T(\theta V_t)^{-1}(x_t - \bar{x}_t) + L_t(Y_t)$ exists at time t, then substitute $\bar{P}_t(x_t, Y_t)$ into (A.1). We see the quadratic form holds also for $\bar{P}_{t+1}(x_{t+1}, Y_{t+1})$. The value $\bar{x}_{t+1}$ is the value of the state which extremizes $\bar{P}_{t+1}(x_{t+1}, Y_{t+1})$. We know by assumption at $t = 0$, the quadratic form $\bar{P}_0(x_0, Y_0) = (x_0 - \bar{x}_0)^T(\theta V_0)^{-1}(x_0 - \bar{x}_0)$ exists.

Writing down the stationary condition of (A.1) about x_{t-1} and x_t, where $\bar{x}_{t+1}$ and x_t^* are the values extremizing $\bar{P}_{t+1}(x_{t+1}, Y_{t+1})$, results in

$$W_t^{-1}(x_{t+1} - A_t x_t - B_t u_t) = 0 \quad \text{(A.2)}$$

$$- A_t^T W_t^{-1}(x_{t+1} - A_t x_t - B_t u_t) + \theta Q_t x_t + V_t^{-1}(x_t - \bar{x}_t) - H_t^T \Theta_t^{-1}(z_t - H_t x_t) = 0 \quad \text{(A.3)}$$

Substitute (A.2) into (A.3), the first term in (A.3) will be zero. So from (A.3), we obtain

$$x_t^* = (V_t^{-1} + \theta Q_t + H_t^T \Theta_t^{-1} H_t)^{-1} V_t^{-1} \bar{x}_t + (V_t^{-1} + \theta Q_t + H_t^T \Theta_t^{-1} H_t)^{-1} H_t^T \Theta_t^{-1} z_t \quad \text{(A.4)}$$

From (A.2)(A.4)

$$\bar{x}_{t+1} = A_t x_t + B_t u_t$$

$$\bar{x}_{t+1} = A_t(V_t^{-1} + \theta Q_t + H_t^T \Theta_t^{-1} H_t)^{-1} V_t^{-1} \bar{x}_t + A_t(V_t^{-1} + \theta Q_t + H_t^T \Theta_t^{-1} H_t)^{-1} H_t^T \Theta_t^{-1} z_t + B_t u_t$$

Rewrite the first term

$$A_t(V_t^{-1} + \theta Q_t + H_t^T \Theta_t^{-1} H_t)^{-1} V_t^{-1} \bar{x}_t$$

$$= A_t(V_t^{-1} + \theta Q_t + H_t^T \Theta_t^{-1} H_t)^{-1}[(V_t^{-1} + \theta Q_t + H_t^T \Theta_t^{-1} H_t) - (\theta Q_t + H_t^T \Theta_t^{-1} H_t)]\bar{x}_t$$

$$= A_t \bar{x}_t - A_t(\theta Q_t + H_t^T \Theta_t^{-1} H_t + V_t^{-1})^{-1}(\theta Q_t + H_t^T \Theta_t^{-1} H_t)\bar{x}_t$$

Therefore, (3.19) is obtained. The following matrix is associated with the quadratic in x_{t+1}, x_t of (A.1)

$$\begin{bmatrix} \beta & \gamma \\ \gamma^T & \delta \end{bmatrix} = \begin{bmatrix} W_t^{-1} & -W_t^{-1} A_t \\ -A_t^T W_t^{-1} & V_t^{-1} + \theta Q_t + H_t^T \Theta_t^{-1} H_t + A_t^T W_t^{-1} A_t \end{bmatrix}$$

If $V_t^{-1} + \theta Q_t + H_t^T \Theta_t^{-1} H_t + A_t^T W_t^{-1} A_t > 0$ ($\theta > 0$ or $\theta < 0$), by extremizing (A.1) w.r.t. x_t, we can define $V_{t+1}^{-1} = \beta - \gamma \delta^{-1} \gamma^T$. Then

$$V_{t+1} = (\beta - \gamma \delta^{-1} \gamma^T)^{-1} = \beta^{-1} + \beta^{-1} \gamma (\delta - \gamma^T \beta^{-1} \gamma)^{-1} \gamma^T \beta^{-1} \tag{A.5}$$

The derivation of $L_{t+1}(Y_{t+1})$ in (3.18) will be given as follows. From the relation $\bar{P}_{t+1}(x_{t+1}, Y_{t+1}) = (x_{t+1} - \bar{x}_{t+1})^T (\theta V_{t+1})^{-1}(x_{t+1} - \bar{x}_{t+1}) + L_{t+1}(Y_{t+1})$, we know

$$L_{t+1}(Y_{t+1}) = \underset{x_{t+1}}{\text{ext}} \, \bar{P}_{t+1}(x_{t+1}, Y_{t+1})$$

$$= \underset{x_{t+1}}{\text{ext}} \, \underset{x_t}{\text{ext}}[\bar{P}_t(x_t, Y_t) + x_t^T Q_t x_t + u_t^T R_t u_t + \theta^{-1}(n_t(x_{t+1}, x_t, u_t) + m_t(z_t, x_t))]$$

Extremizing with respect to x_{t+1} will eliminate the term $n_t(x_{t+1}, x_t, u_t)$,

$$L_{t+1}(Y_{t+1}) = \underset{x_t}{\text{ext}}[(x_t - \bar{x}_t)^T (\theta V_t)^{-1}(x_t - \bar{x}_t)^T + L_t(Y_t) + x_t^T Q_t x_t + u_t^T R_t u_t + (z_t - H_t x_t)^T (\theta \Theta_t)^{-1}(z_t - H_t x_t)^T]$$

$$= \underset{x_t}{\text{ext}}\{x_t^T[(\theta V_t)^{-1} + H_t^T (\theta \Theta_t)^{-1} H_t + Q_t]x_t - 2x_t^T[(\theta V_t)^{-1} \bar{x}_t + H_t^T (\theta \Theta_t)^{-1} z_t]$$

$$+ \bar{x}_t^T (\theta V_t)^{-1} \bar{x}_t + u_t^T R_t u_t + z_t^T (\theta \Theta_t)^{-1} z_t + L_t(Y_t) \tag{A.6}$$

If $V_t^{-1} + H_t \Theta_t^{-1} H_t + \theta Q_t > 0$ for $\theta < 0$ or $\theta > 0$, then the optimal value of x_t for (A.6) is

$$x_t^* = [V_t^{-1} + H_t^T \Theta_t^{-1} H_t + \theta Q_t]^{-1}[V_t^{-1} \bar{x}_t + H_t^T \Theta_t^{-1} z_t] \tag{A.7}$$

Substitute x_t^* back into (A.6), we obtain

$$L_{t+1}(Y_{t+1}) = \bar{x}_t^T (\theta V_t)^{-1} \bar{x}_t + u_t^T R_t u_t + z_t^T (\theta \Theta_t)^{-1} z_t + L_t(Y_t)$$

$$- \theta^{-1}(V_t^{-1} \bar{x}_t + H_t^T \Theta_t^{-1} z_t)^T (V_t^{-1} + H_t^T \Theta_t^{-1} H_t^T + \theta Q_t)^{-1}(V_t^{-1} \bar{x}_t + H_t^T \Theta_t^{-1} z_t) \tag{A.8}$$

In order to obtain V_{t+1}(A.5) and x_t^*(A.7), it is assumed that $V_t^{-1} + \theta Q_t + H_t^T \Theta_t^{-1} H_t + A_t^T W_t^{-1} A_t > 0$ and $V_t^{-1} + \theta Q_t + H_t^T \Theta_t^{-1} H_t > 0$ when $\theta < 0$ or $\theta > 0$. The following relation will be proved later in Lemma 3.1 that $V_t^{-1} + H_t^T \Theta_t^{-1} H_t + \theta Q_t > 0$ if and only if $V_t^{-1} + H_t^T \Theta_t^{-1} H_t + \theta Q_t + A_t^T W_t^{-1} A_t > 0$, and in turn the inequality $V_t > 0$ exist. Notice when $\theta > 0$, the above two inequalities always exist.

B

Proof of Lemma 3.1

From the matrix inverse identity

$$Q^{-1} - Q^{-1} A [R + A^T Q^{-1} A]^{-1} A^T Q^{-1} = (Q + A R^{-1} A^T)^{-1} \tag{B.1}$$

and (3.17)

$$
\begin{aligned}
V_{t+1} &= W_t + A_t[(V_t^{-1} + \theta Q_t + A_t^T W_t^{-1} A_t + H_t^T \Theta_t^{-1} H_t) - A_t^T W_t^{-1} A_t]^{-1} A_t^T \\
&= W_t[W_t^{-1} - W_t^{-1} A_t (A_t^T W_t^{-1} A_t - (V_t^{-1} + \theta Q_t + A_t^T W_t^{-1} A_t + H_t^T \Theta_t^{-1} H_t))^{-1} A_t^T W_t^{-1}] W_t \\
&= W_t[W_t - A_t(V_t^{-1} + \theta Q_t + A_t^T W_t^{-1} A_t + H_t^T \Theta_t^{-1} H_t)^{-1} A_t^T]^{-1} W_t \\
&= [W_t^{-1} - W_t^{-1} A_t (V_t^{-1} + \theta Q_t + A_t^T W_t^{-1} A_t + H_t^T \Theta_t^{-1} H_t)^{-1} A_t^T W_t^{-1}]^{-1}
\end{aligned}
$$

$$W_t^{-1} - V_{t+1}^{-1} = W_t^{-1} A_t (V_t^{-1} + \theta Q_t + A_t^T W_t^{-1} A_t + H_t^T \Theta_t^{-1} H_t)^{-1} A_t^T W_t^{-1} \tag{B.2}$$

If (3.21) exists, then

$$W_t^{-1} - V_{t+1}^{-1} \geq 0 \Rightarrow V_{t+1} \geq W_t > 0 \Rightarrow V_{t+1} > 0 \tag{B.3}$$

From (3.17)

$$A_t(V_t^{-1} + \theta Q_t + H_t^T \Theta_t^{-1} H_t)^{-1} A_t^T = V_{t+1} - W_t \geq 0 \tag{B.4}$$

Because $V_t^{-1} + \theta Q_t + H_t^T \Theta_t^{-1} H_t$ is assumed nonsingular, therefore (3.20) is obtained.

$$V_t^{-1} + \theta Q_t + H_t^T \Theta_t^{-1} H_t > 0$$

If (3.20) exists, reverse the above derivation (B.4)-(B.2), and the inequality (3.21) must be true. Or by adding $A_t^T W_t^{-1} A_t \geq 0$ to (3.20), the inequality (3.21) is obtained. From (3.17) and (3.20), (B.3) is obtained and V_t is positive definite.

C

First, a three-member team problem is solved. Then, it is only an easy exercise to solve the M-member team problem. To solve the three-member team problem, we formulate as in [19] or chapter 3 of [9] a person-by-person optimization problem. For the i-th member $i \in \{1,2,3\}$, each of the other two control functions is at its one-person optimum problem. Substitute $u^j = D^j z^j + C^j \bar{x}$, $j \neq i$ and $i,j \in \{1,2,3\}$ into the function f of (4.5)

$$
\min_{u^i} \text{ext}_{\hat{z}(i)} [\bar{x}^T Q_{11} \bar{x} + 2\bar{x}^T Q_{12}^i z^i + \sum_{j=1,j\neq i}^{3} 2\bar{x}^T Q_{12}^j z^j + 2\bar{x}^T Q_{13}^i u^i + 2\bar{x}^T [\sum_{j=1,j\neq i}^{3} Q_{13}^j (D^j z^j + C^j \bar{x})]
$$

$$
+(z^i)^T Q_{22}^{ii} z^i + \sum_{j=1,j\neq i}^{3} 2(z^i)^T Q_{22}^{ij} z^j + \sum_{k=1,k\neq i}^{3} (z^k)^T [\sum_{j=1,j\neq i}^{3} Q_{22}^{kj} z^j]_{j\neq i,k\neq i} + (z^i)^T Q_{23}^{ii} u^i
$$

$$
+2(z^i)^T [\sum_{j=1,j\neq i}^{3} Q_{23}^{ij} (D^j z^j + C^j \bar{x})] + \sum_{k=1,k\neq i}^{3} (z^k)^T [\sum_{j=1,j\neq i}^{3} Q_{23}^{kj} (D^j z^j + C^j \bar{x})] + (u^i)^T Q_{33}^{ii} u^i
$$

$$
+2(u^i)^T [\sum_{j=1,j\neq i}^{3} Q_{33}^{ij} (D^j z^j + C^j \bar{x})] + \sum_{k=1,k\neq i}^{3} (D^k z^k + C^k \bar{x})^T [\sum_{j=1,j\neq i}^{3} Q_{33}^{kj} (D^j z^j + C^j \bar{x})]_{j\neq i,k\neq i}
$$

The stationary conditions with respect to u^i and $\hat{z}(i)$ yield

$$
Q_{33}^{ii} u^i + [(Q_{13}^i)^T + \sum_{j=1,j\neq i}^{3} Q_{33}^{ij} C^j] \bar{x} + (Q_{23}^{ii})^T z^i + \beta^{iT} \hat{z}(i) = 0 \tag{C.1}
$$

$$
[(\alpha^i)^{-1} \hat{z}(i) + \delta^i \bar{x} + \gamma^i z^i + \beta^i u^i] = 0 \tag{C.2}
$$

By solving $\hat{z}(i)$ explicitly from (C.2) and substituting into (C.1), (4.8) and (4.9) are obtained where $M = 3$ and

$$
(\alpha^1)^{-1} = \begin{bmatrix} D^{2T} Q_{33}^{22} D^2 + Q_{23}^{22} D^2 + (Q_{23}^{22} D^2)^T + Q_{22}^{22} & D^{2T} Q_{33}^{23} D^3 + Q_{23}^{23} D^3 + (Q_{23}^{32} D^2)^T + Q_{22}^{23} \\ D^{3T} Q_{33}^{32} D^2 + Q_{23}^{32} D^2 + (Q_{23}^{23} D^3)^T + Q_{22}^{32} & D^{3T} Q_{33}^{33} D^3 + Q_{23}^{33} D^3 + (Q_{23}^{33} D^3)^T + Q_{22}^{33} \end{bmatrix}
$$

$$
(\alpha^2)^{-1} = \begin{bmatrix} D^{1T} Q_{33}^{11} D^1 + Q_{23}^{11} D^1 + (Q_{23}^{11} D^1)^T + _{\iota,22}^1 & D^{1T} Q_{33}^{13} D^3 + Q_{23}^{13} D^3 + (Q_{23}^{31} D^1)^T + Q_{22}^{13} \\ D^{3T} Q_{33}^{31} D^1 + Q_{23}^{31} D^1 + (Q_{23}^{13} D^3)^T + Q_{22}^{31} & D^{3T} Q_{33}^{33} D^3 + Q_{23}^{33} D^3 + (Q_{23}^{33} D^3)^T + Q_{22}^{33} \end{bmatrix}
$$

$$
(\alpha^3)^{-1} = \begin{bmatrix} D^{1T} Q_{33}^{11} D^1 + Q_{23}^{11} D^1 + (Q_{23}^{11} D^1)^T + Q_{22}^{11} & D^{1T} Q_{33}^{12} D^2 + Q_{23}^{12} D^2 + (Q_{23}^{21} D^1)^T + Q_{22}^{12} \\ D^{2T} Q_{33}^{21} D^1 + Q_{23}^{21} D^1 + (Q_{23}^{12} D^2)^T + Q_{22}^{21} & D^{2T} Q_{33}^{22} D^2 + Q_{23}^{22} D^2 + (Q_{23}^{22} D^2)^T + Q_{22}^{22} \end{bmatrix} \tag{C.3}
$$

$$
\beta^1 = \begin{bmatrix} Q_{23}^{21} + D^{2T} Q_{33}^{21} \\ Q_{23}^{31} + D^{3T} Q_{33}^{31} \end{bmatrix} \quad \beta^2 = \begin{bmatrix} Q_{23}^{12} + D^{1T} Q_{33}^{12} \\ Q_{23}^{32} + D^{3T} Q_{33}^{32} \end{bmatrix} \quad \beta^3 = \begin{bmatrix} Q_{23}^{13} + D^{1T} Q_{33}^{13} \\ Q_{23}^{23} + D^{2T} Q_{33}^{23} \end{bmatrix} \tag{C.4}
$$

$$\gamma^1 = \begin{bmatrix} Q_{22}^{21} + (Q_{23}^{12}D^2)^T \\ Q_{22}^{31} + (Q_{23}^{13}D^3)^T \end{bmatrix} \quad \gamma^2 = \begin{bmatrix} Q_{22}^{12} + (Q_{23}^{21}D^1)^T \\ Q_{22}^{32} + (Q_{23}^{23}D^3)^T \end{bmatrix} \quad \gamma^3 = \begin{bmatrix} Q_{22}^{13} + (Q_{23}^{31}D^1)^T \\ Q_{22}^{23} + (Q_{23}^{32}D^2)^T \end{bmatrix} \quad \text{(C.5)}$$

$$\delta^1 = \begin{bmatrix} (Q_{12}^2 + Q_{13}^2 D^2)^T + Q_{23}^{22}C^2 + Q_{23}^{23}C^3 + D^{2T}(Q_{33}^{22}C^2 + Q_{33}^{23}C^3) \\ (Q_{12}^3 + Q_{13}^3 D^3)^T + Q_{23}^{32}C^2 + Q_{23}^{33}C^3 + D^{3T}(Q_{33}^{32}C^2 + Q_{33}^{33}C^3) \end{bmatrix}$$

$$\delta^2 = \begin{bmatrix} (Q_{12}^1 + Q_{13}^1 D^1)^T + Q_{23}^{11}C^1 + Q_{23}^{13}C^3 + D^{1T}(Q_{33}^{11}C^1 + Q_{33}^{13}C^3) \\ (Q_{12}^3 + Q_{13}^3 D^3)^T + Q_{23}^{31}C^1 + Q_{23}^{33}C^3 + D^{3T}(Q_{33}^{31}C^1 + Q_{33}^{33}C^3) \end{bmatrix} \quad \text{(C.6)}$$

$$\delta^3 = \begin{bmatrix} (Q_{12}^1 + Q_{13}^1 D^1)^T + Q_{23}^{11}C^1 + Q_{23}^{12}C^2 + D^{1T}(Q_{33}^{11}C^1 + Q_{33}^{12}C^2) \\ (Q_{12}^2 + Q_{13}^2 D^2)^T + Q_{23}^{21}C^1 + Q_{23}^{22}C^2 + D^{2T}(Q_{33}^{21}C^1 + Q_{33}^{22}C^2) \end{bmatrix}$$

Next we will discuss second-order sufficiency. By observing (C.1) and (C.2), if $\alpha^i > 0(\theta > 0)$, $\alpha^i < 0(\theta < 0)$ and $Q_{33}^{ii} > 0$, then the control gain D^i of (4.8) and C^i of (4.9) for person-by-person optimization can be solved. The assumption $\bar{R} > 0(\theta > 0)$ and $\bar{R} < 0(\theta < 0)$ implies $\alpha^i > 0(\theta > 0)$, $\alpha^i < 0(\theta < 0)$, where α^i is a minor of $\bar{R}$.

The optimal cost $f(u^*, z^*, \bar{x})$ in (4.10) is to be determined. We substitute the optimal team control function $u^* = C\bar{x} + Dz$ into the function f in (4.5), where C and D are given in (4.8) and (4.9)

$$f(u^*, z^*, \bar{x}) = \underset{z}{\text{ext}}[\bar{x}^T Q_{11}\bar{x} + 2\bar{x}^T Q_{12}z + 2\bar{x}^T Q_{13}u + z^T Q_{22}z + 2z^T Q_{23}u + u^T Q_{33}u]$$

$$= \underset{z}{\text{ext}}[z^T(D^T Q_{33}D + Q_{23}D + (Q_{23}D)^T + Q_{22})z + 2\bar{x}^T(C^T Q_{33}D + Q_{13}D + (Q_{23}C)^T + Q_{12})z$$

$$+ \bar{x}^T(C^T Q_{33}C + Q_{13}C + (Q_{13}C)^T + Q_{11})\bar{x}] \quad \text{(C.7)}$$

Define $\bar{R}$ and $\bar{\delta}$ as in (4.6) and (4.7) respectively. Then, the stationary condition yields $\bar{R}z + \bar{\delta}\bar{x} = 0$.

Note that "ext" means "min" for $\theta > 0$ and "max" for $\theta < 0$. Assume $\bar{R} > 0$ when $\theta > 0$ or $\bar{R} < 0$ when $\theta < 0$. Then, the equation $z = -\bar{R}^{-1}\bar{\delta}\bar{x}$ optimizes the function in (C.7) and (4.10) is obtained. If the assumption $\bar{R} > 0(\theta > 0)$ or $\bar{R} < 0(\theta < 0)$ is not satisfied, then the expected value of the cost function will be infinite or negative infinite. (See equations (5.2) to (5.3))

D

A more detailed derivation about (5.5),(5.6) and (5.7) is given in this appendix. Let $Y = [H_t^T(\theta\Theta_t)^{-1}H_t + (\theta V_t)^{-1} + \Pi_t]^{-1}$ and substitute (3.24) into (5.4)

$$\underset{z_t}{\text{ext}}\{-[(\theta V_t)^{-1}\bar{x}_t + H_t^T(\theta\Theta_t)^{-1}z_t]^T Y[(\theta V_t)^{-1}\bar{x}_t + H_t^T(\theta\Theta_t)^{-1}z_t]$$

$$+ z_t^T(\theta\Theta_t)^{-1}z_t + \bar{x}_t^T(\theta V_t)^{-1}\bar{x}_t\}$$

$$= \underset{z_t}{\text{ext}}\{z_t^T[(\theta\Theta_t)^{-1} - (\theta\Theta_t)^{-1}H_t Y H_t^T(\theta\Theta_t)^{-1}]z_t$$

$$- 2\bar{x}_t^T(\theta V_t)^{-1}Y H_t^T(\theta\Theta_t)^{-1}z_t - \bar{x}_t^T(\theta V_t)^{-1}Y(\theta V_t)^{-1}\bar{x}_t + \bar{x}_t^T(\theta V_t)^{-1}\bar{x}_t\} \quad \text{(D.1)}$$

By using (D.1) and the matrix inverse identity (B.1), (5.5) is obtained. The first-order stationary condition of z_t in (5.5) is

$$z_t^* = [\theta\Theta_t + H_t((\theta V_t)^{-1} + \Pi_t)^{-1}H_t^T](\theta\Theta_t)^{-1}H_t Y(\theta V_t)^{-1}\bar{x}_t \qquad (D.2)$$

and z_t^* can be simplified as

$$z_t^* = \{H_t + H_t[(\theta V_t)^{-1} + \Pi_t]^{-1}H_t^T(\theta\Theta_t)^{-1}H_t\}Y(\theta V_t)^{-1}\bar{x}_t$$

$$= H_t[(\theta V_t)^{-1} + \Pi_t]^{-1}Y^{-1}Y(\theta V_t)^{-1}\bar{x}_t$$

$$= H_t(I + \theta V_t\Pi_t)^{-1}\bar{x}_t$$

The last equation is (5.6). Substitute z_t^* in (D.2) into (5.5)

$$-\bar{x}^T\{(\theta V_t)^{-1}YH_t^T(\theta\Theta_t)^{-1}\}\{\theta\Theta_t + H_t((\theta V_t)^{-1} + \Pi_t)^{-1}H_t^T\}\{(\theta\Theta_t)^{-1}H_t Y(\theta V_t)^{-1}\bar{x}_t\}$$

$$-\bar{x}_t^T(\theta V_t)^{-1}Y(\theta V_t)^{-1}\bar{x}_t + \bar{x}_t^T(\theta V_t)^{-1}\bar{x}_t \qquad (D.3)$$

Let

$$Z = H_t^T(\theta\Theta_t)^{-1}\{\theta\Theta_t + H_t[(\theta V_t)^{-1} + \Pi_t]^{-1}H_t^T\}(\theta\Theta_t)^{-1}H_t$$

$$= H_t^T(\theta\Theta_t)^{-1}H_t + H_t^T(\theta\Theta_t)^{-1}H_t[(\theta V_t)^{-1} + \Pi_t]^{-1}H_t^T(\theta\Theta_t)^{-1}H_t$$

$$= H_t^T(\theta\Theta_t)^{-1}H_t - H_t^T(\theta\Theta_t)^{-1}H_t\{H_t^T(\theta\Theta_t)^{-1}H_t - Y^{-1}\}^{-1}H_t^T(\theta\Theta_t)^{-1}H_t$$

By using (B.1)

$$Z = [(H_t^T(\theta\Theta_t)^{-1}H_t)^{-1} - Y]^{-1},$$

then (5.7) is obtained from (D.3).

$$-\bar{x}_t^T(\theta V_t)^{-1}\{Y[(H_t^T(\theta\Theta_t)^{-1}H_t)^{-1} - Y]^{-1}Y\}(\theta V_t)^{-1}\bar{x}_t$$

$$-\bar{x}_t^T(\theta V_t)^{-1}Y(\theta V_t)^{-1}\bar{x}_t + \bar{x}_t^T(\theta V_t)^{-1}\bar{x}_t$$

$$= -\bar{x}_t^T(\theta V_t)^{-1}[Y^{-1} - H_t^T(\theta\Theta_t)^{-1}H_t]^{-1}(\theta V_t)^{-1}\bar{x}_t + \bar{x}_t^T(\theta V_t)^{-1}\bar{x}_t$$

$$= -\bar{x}_t^T(\theta V_t)^{-1}[(\theta V_t)^{-1} + \Pi_t]^{-1}(\theta V_t)^{-1}\bar{x}_t^T + \bar{x}_t^T(\theta V_t)^{-1}\bar{x}_t$$

$$= \bar{x}_t^T[\theta V_t + \Pi_t^{-1}]^{-1}\bar{x}_t = \bar{x}_t^T\Pi_t[\theta V_t\Pi_t + I]^{-1}\bar{x}_t$$

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